Fresnel Equations and EM Power Flow

Reading - Shen and Kong – Ch. 4

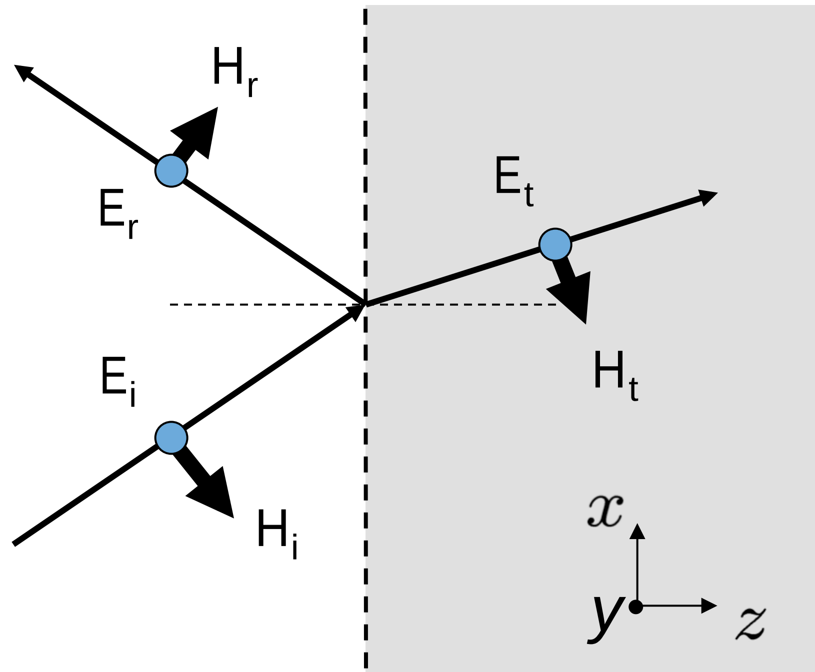
<u>Outline</u>

- Review of Oblique Incidence
- Review of Snell's Law
- Fresnel Equations
- Evanescence and TIR
- Brewster's Angle
- EM Power Flow

TRUE / FALSE

1. This EM wave is TE (transverse electric) polarized:

____



2. Snell's Law only works for TE polarized light. ____

3. Total internal reflection only occurs when light goes from a high index material to a low index material. ____

Refraction

Water Waves

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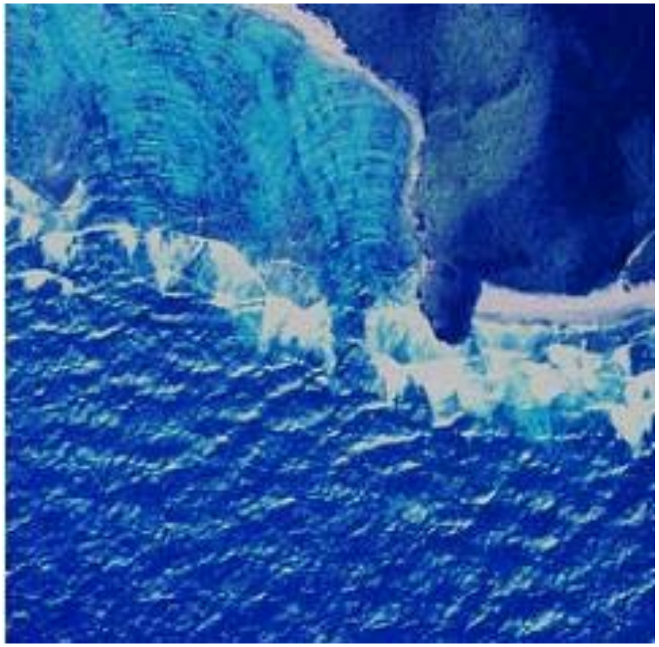
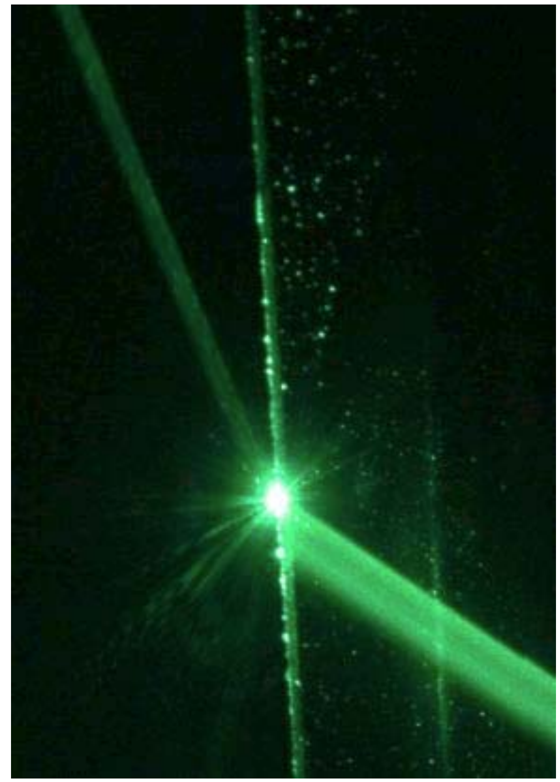


Waves refract at the top where the water is shallower

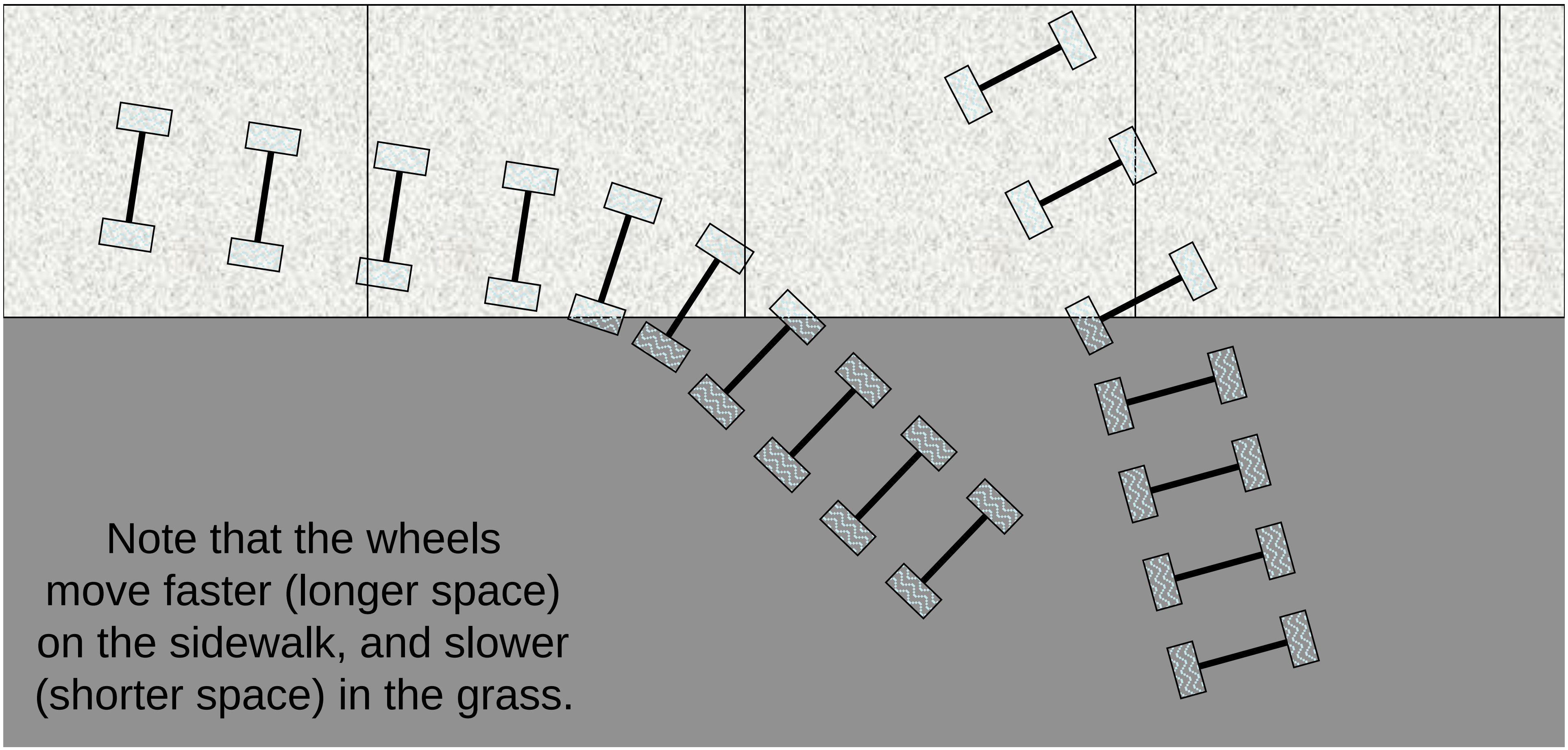
E&M Waves



Refraction involves a change in the direction of wave propagation due to a change in propagation speed. It involves the oblique incidence of waves on media boundaries, and hence wave propagation in at least two dimensions.

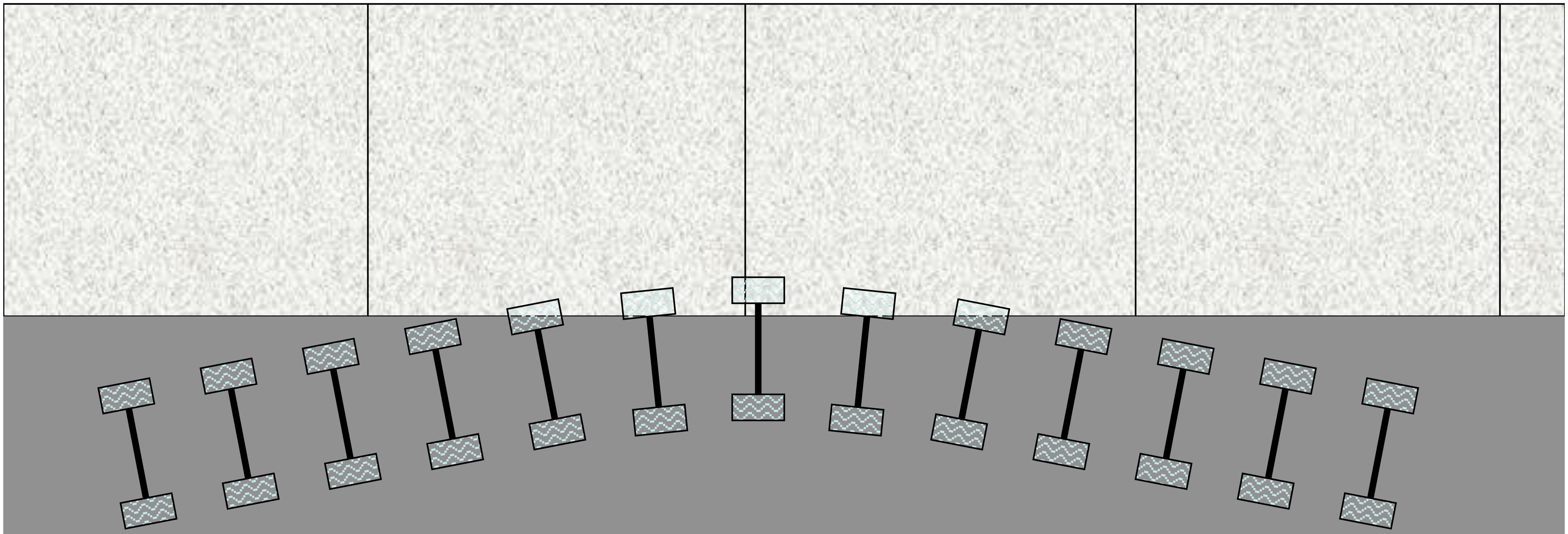
Refraction in Suburbia

Think of refraction as a pair of wheels on an axle going from a sidewalk onto grass. The wheel in the grass moves slower, so the direction of the wheel pair changes.



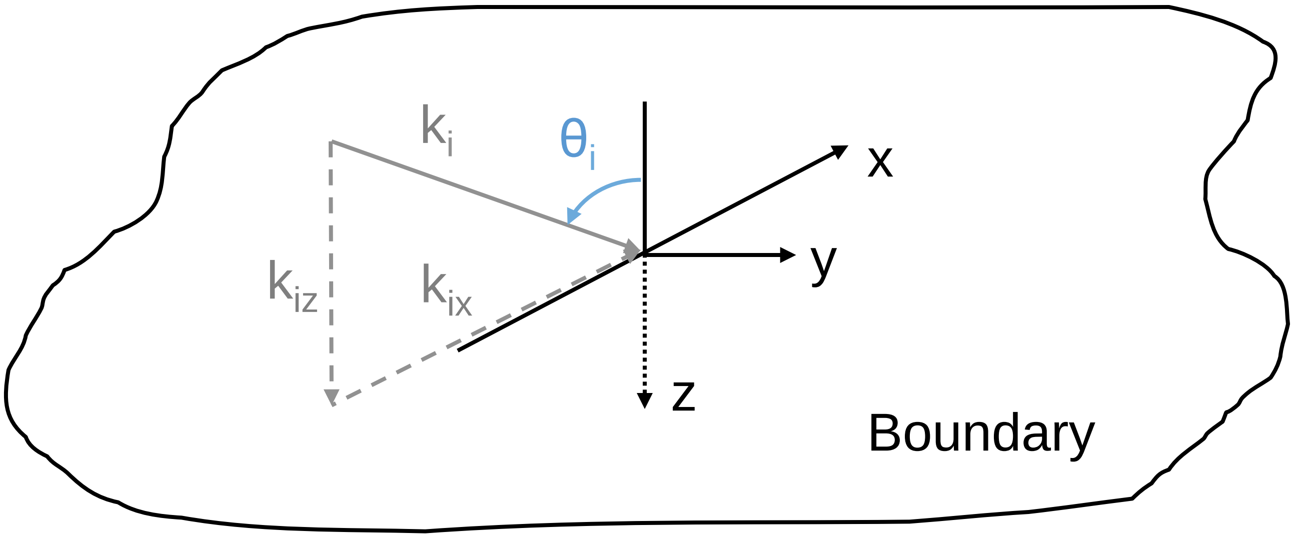
Total Internal Reflection in Suburbia

Moreover, this wheel analogy is mathematically equivalent to the refraction phenomenon. One can recover Snell's law from it: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.



The upper wheel hits the sidewalk and starts to go faster, which turns the axle until the upper wheel re-enters the grass and wheel pair goes straight again.

Oblique Incidence (3D view)

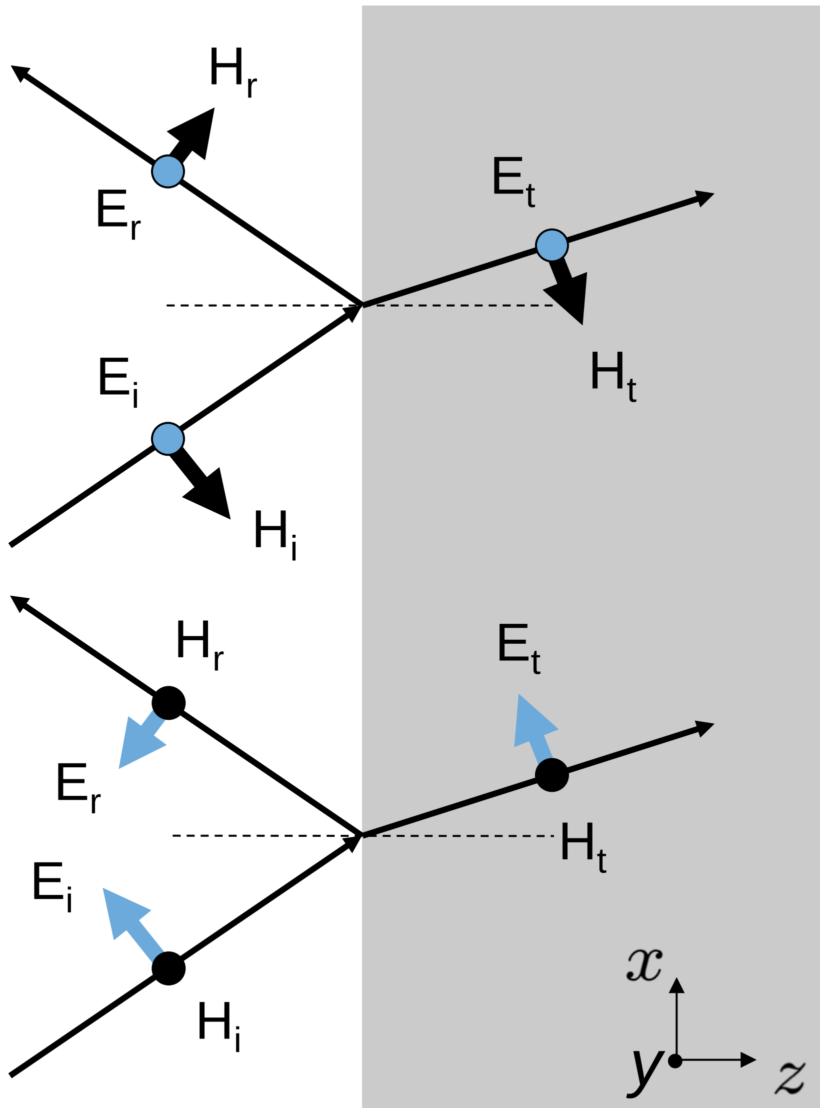


$k_{ix} = \quad k_i \sin(\theta_i)$

$k_{iz} = \quad k_i \cos(\theta_i)$

Identical definitions for k_r and k_t

Oblique Incidence at Dielectric Interface

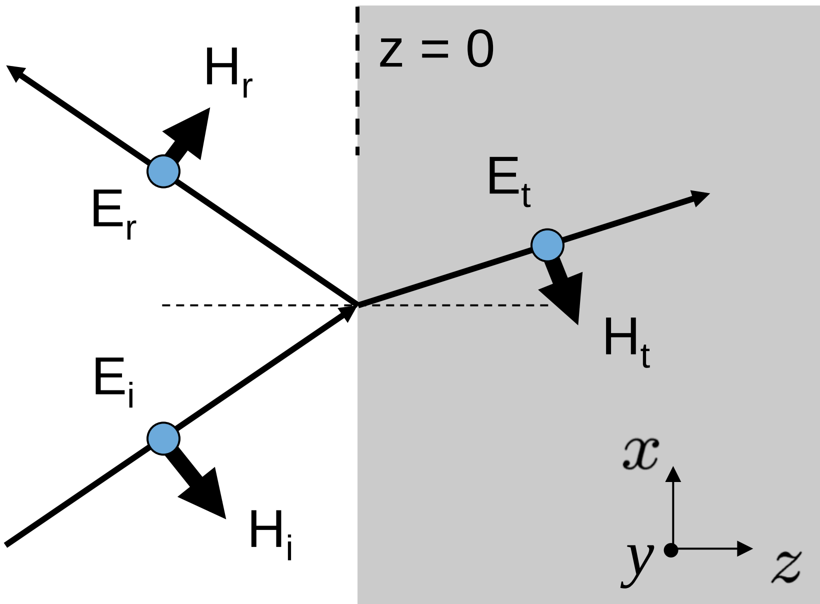


Transverse
Electric Field

Transverse
Magnetic Field

Why do we consider only
these two polarizations?

Partial TE Analysis



$$\vec{E} = \vec{E}_o e^{j(\omega t - \vec{k}\cdot\vec{r})}$$

$$\omega_i = \omega_r = \omega_t$$

$$\vec{E}_i = \hat{y} E_o^i e^{-jk_{ix}x - jk_{iz}z}$$

$$\vec{E}_r = \hat{y} E_o^r e^{-jk_{rx}x + jk_{rz}z}$$

$$\vec{E}_t = \hat{y} E_o^t e^{-jk_{tx}x - jk_{tz}z}$$

Tangential E must be continuous at the boundary z = 0 for all x and for t.

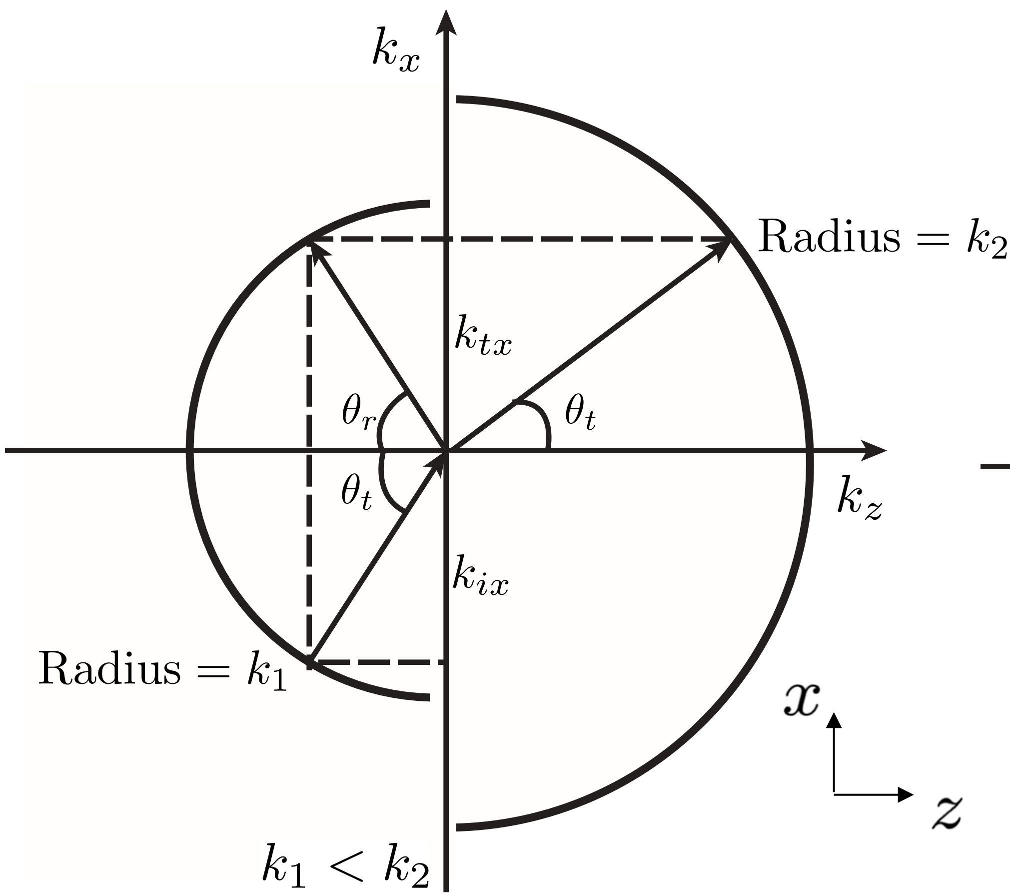
$$E_o^i e^{-jk_{ix}} + E_o^r e^{-jk_{rx}x} = E_o^t e^{-jk_{tx}x}$$

This is possible if and only if $k_{ix} = k_{rx} = k_{tx}$ and $\omega_i = \omega_r = \omega_t$. The former condition is phase matching.

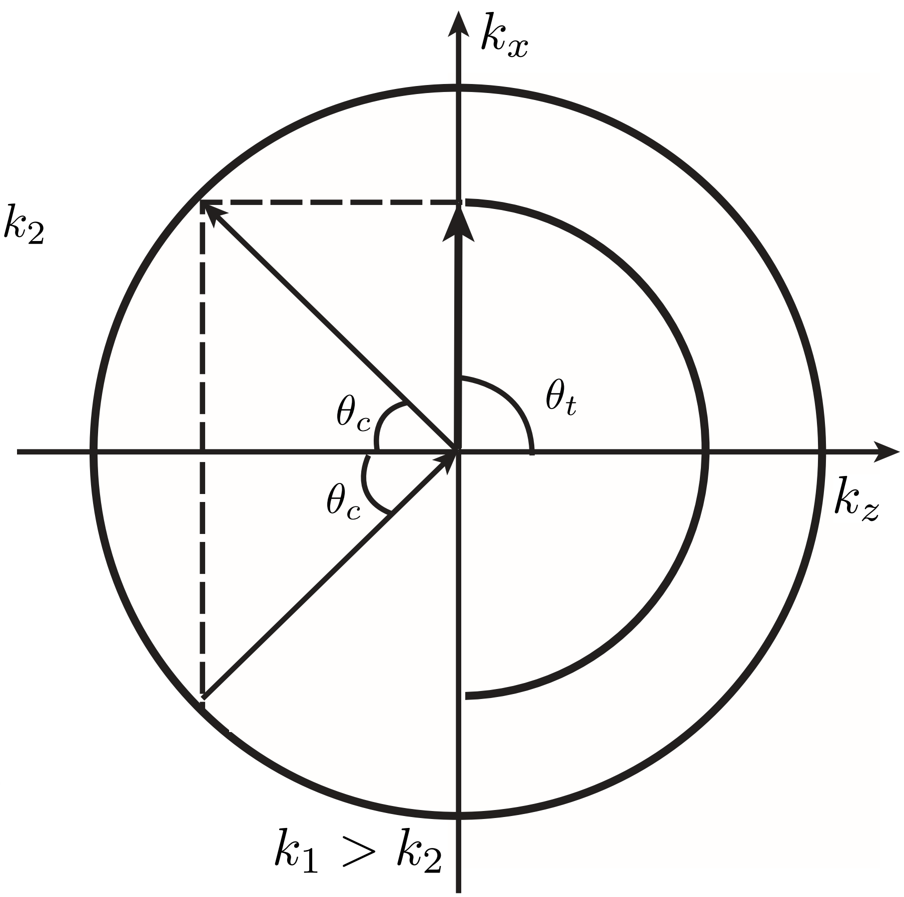
Snell's Law Diagram

Tangential E field is continuous … $k_{ix} = k_{tx}$

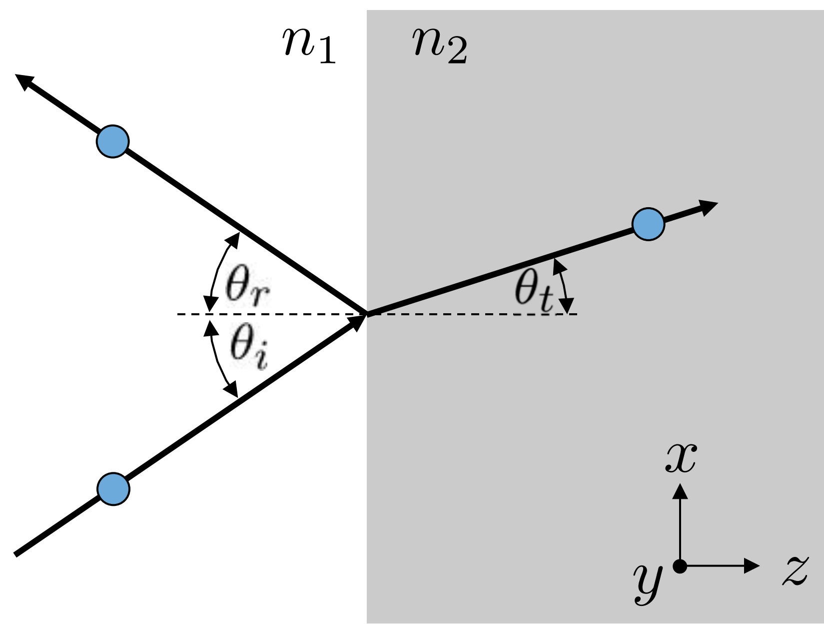
Refraction



Total Internal Reflection



Snell's Law



$$kix = k_{rx}$$

$$n_1 \sin\theta_i = n_1 \sin\theta_r$$

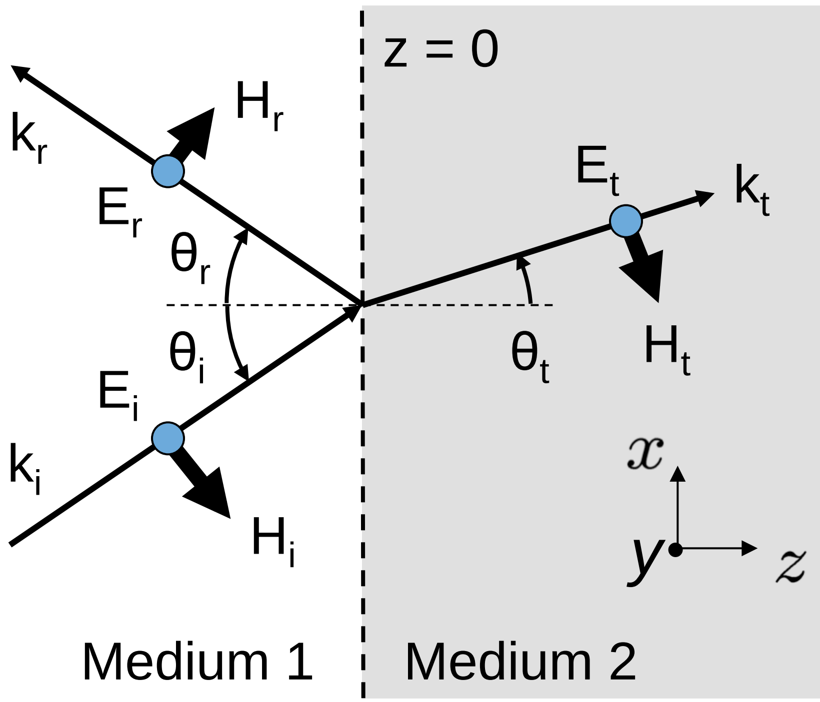
$$\theta_i = \theta_r$$

$$kix = k_{tx}$$

$$n_2 \sin\theta_i = n_2 \sin\theta_t$$

Snell's Law

TE Analysis - Set Up



$$k_x^2 + k_z^2 = k^2 = \omega^2 \mu \epsilon$$

$$k_x = k \sin\theta$$

$$k_z = k \cos\theta$$

$$\vec{E}_i = \hat{y} E_o e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{E}_r = \hat{y} r E_o e^{j(-k_{ix}x + k_{iz}z)}$$

$$\vec{E}_t = \hat{y} t E_o e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \mu \vec{H}}{\partial t}$$

$$-jk \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H} = \frac{1}{\omega\mu} k \times \vec{E}$$

To get H, use Faraday's Law

$$\vec{H}_i = (\hat{z}k_{ix} - \hat{x}k_{iz}) \frac{E_o}{\omega\mu_1} e^{j(-k_{ix}x - k_{iz}z)}$$

$$\vec{H}_r = (\hat{z}k_{ix} + \hat{x}k_{iz}) \frac{rE_o}{\omega\mu_1} e^{j(-k_{ix}x + k_{iz}z)}$$

$$\vec{H}_t = (\hat{z}k_{tx} + \hat{x}k_{tz}) \frac{tE_o}{\omega\mu_2} e^{j(-k_{tx}x - k_{tz}z)}$$

TE Analysis – Boundary Conditions

Incident Wavenumber: $k_i^2 = k_r^2 = \omega^2 \mu \epsilon$

Phase Matching:

$$k_{ix} = k_{rx} = k_{tx}$$

$$\theta_r = \theta_i$$

$$k_i \sin\theta_i = k_t \sin\theta_t$$

Tangential E: $1 + r = t$

Normal μH: $1 + r = t$

Tangential H:

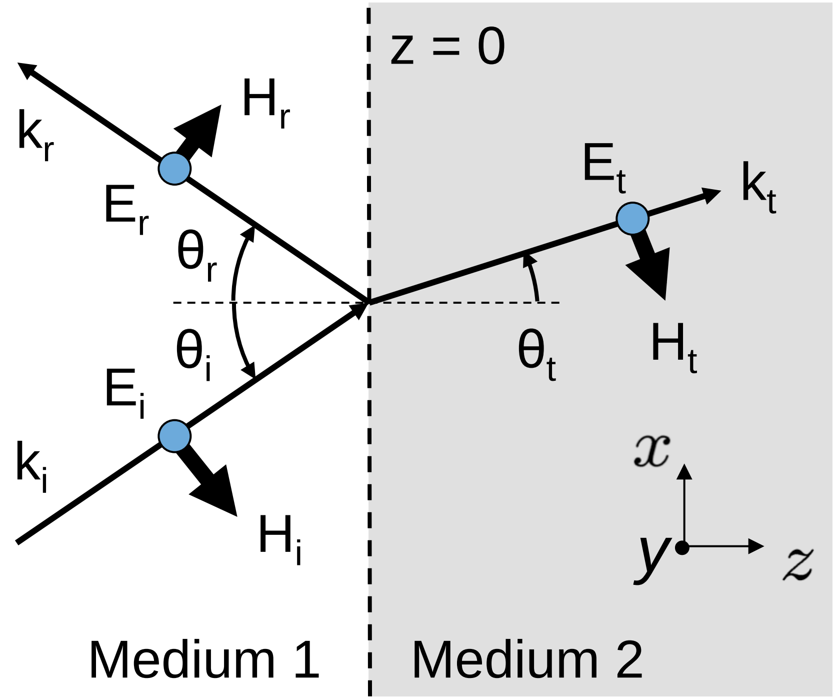
$$\frac{E_o}{\omega\mu_i}(1-r)k_{itz} = \frac{tE_o}{\omega\mu_t}k_{itz}$$

$$(1-r) = t\frac{\sqrt{\mu_t\varepsilon_t}}{\sqrt{\mu_i\varepsilon_i}}\frac{\mu_i}{\mu_t}\frac{\cos\theta_t}{\cos\theta_i} = t\frac{\eta_i}{\eta_t}\frac{\cos\theta_t}{\cos\theta_i}$$

Solution Boundary conditions are:

$$1 + r = t$$

$$1 - r = t\frac{\eta_i}{\eta_t}\frac{\cos\theta_t}{\cos\theta_i}$$



$$t = \frac{2\eta_t\cos\theta_i}{\eta_t\cos\theta_i + \eta_i\cos\theta_t}$$

$$r = \frac{\eta_t\cos\theta_i - \eta_i\cos\theta_t}{\eta_t\cos\theta_i + \eta_i\cos\theta_t}$$

$$E_t = tE_i$$

$$E_r = rE_i$$

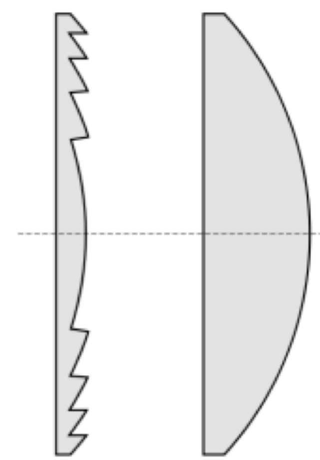
Fresnel Equations

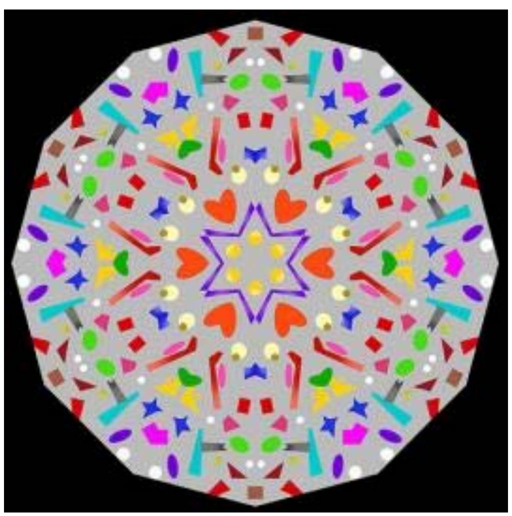
Today's Culture Moment

Sir David Brewster

- Scottish scientist
- Studied at University of Edinburgh at age 12
- Independently discovered Fresnel lens
- Editor of *Edinburgh Encyclopedia and contributor to Encyclopedia Britannica (7th and 8th editions)*
- Inventor of the Kaleidoscope
- Nominated (1849) to the National Institute of France.

Fresnel Lens





Kaleidoscope

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TM Case is the dual of TE

E Gauss: $k \cdot E = 0$

H Gauss: $k \cdot H = 0$

Faraday: $k \times E = \omega\mu H$

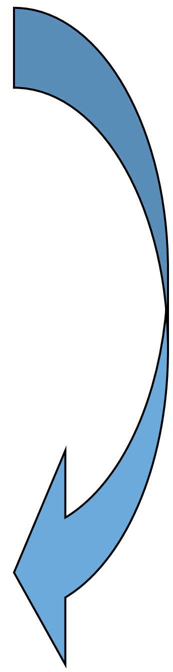
Ampere: $k \times H = -\omega\varepsilon E$

E Gauss: $k \cdot E = 0$

H Gauss: $k \cdot H = 0$

Faraday: $k \times E = \omega\mu H$

Ampere: $k \times H = -\omega\varepsilon E$



$E \rightarrow H$

$H \rightarrow -E$

$\varepsilon \rightarrow \mu$

$\mu \rightarrow \varepsilon$

The TM solution can be recovered from the TE solution.
So, consider only the TE solution in detail.

TE & TM Analysis – Solution

TE solution comes directly from the boundary condition analysis

$$t = \frac{2\eta_t \cos\theta_i}{\eta_t \cos\theta_i + \eta_i \cos\theta_t} \qquad r = \frac{\eta_t \cos\theta_i - \eta_i \cos\theta_t}{\eta_t \cos\theta_i + \eta_i \cos\theta_t}$$

TM solution comes from ε ↔ μ

$$r = \frac{\eta_t^{-1} \cos\theta_i - \eta_i^{-1} \cos\theta_t}{\eta_t^{-1} \cos\theta_i + \eta_i^{-1} \cos\theta_t} \qquad t = \frac{2\eta_t^{-1} \cos\theta_i}{\eta_t^{-1} \cos\theta_i + \eta_i^{-1} \cos\theta_t}$$

Note that the TM solution provides the reflection and transmission coefficients for H, since TM is the dual of TE.

Fresnel Equations - Summary

TE-polarization

$$r_E = \frac{E_o^r}{E_o^i} = \frac{N_i - N_t}{N_i + N_t}$$

$$t_E = \frac{E_o^t}{E_o^i} = \frac{2N_i}{N_i + N_t}$$

$$N_i = \frac{1}{\eta_1} \cos\theta_i$$

$$N_t = \frac{1}{\eta_2} \cos\theta_t$$

TM Polarization

$$r_H = \frac{H_o^r}{H_o^i} = \frac{M_i - M_t}{M_i + M_t}$$

$$t_H = \frac{H_o^t}{H_o^i} = \frac{2M_i}{M_i + M_t}$$

$$M_i = \eta_1 \cos\theta_i$$

$$M_t = \eta_2 \cos\theta_t$$

Fresnel Equations - Summary

From Shen and Kong … just another way of writing the same results

TE Polarization

$$r_{TE} = \frac{E_o^r}{E_o^i} = \frac{\mu_2 k_{iz} - \mu_1 k_{tz}}{\mu_2 k_{iz} + \mu_1 k_{tz}}$$

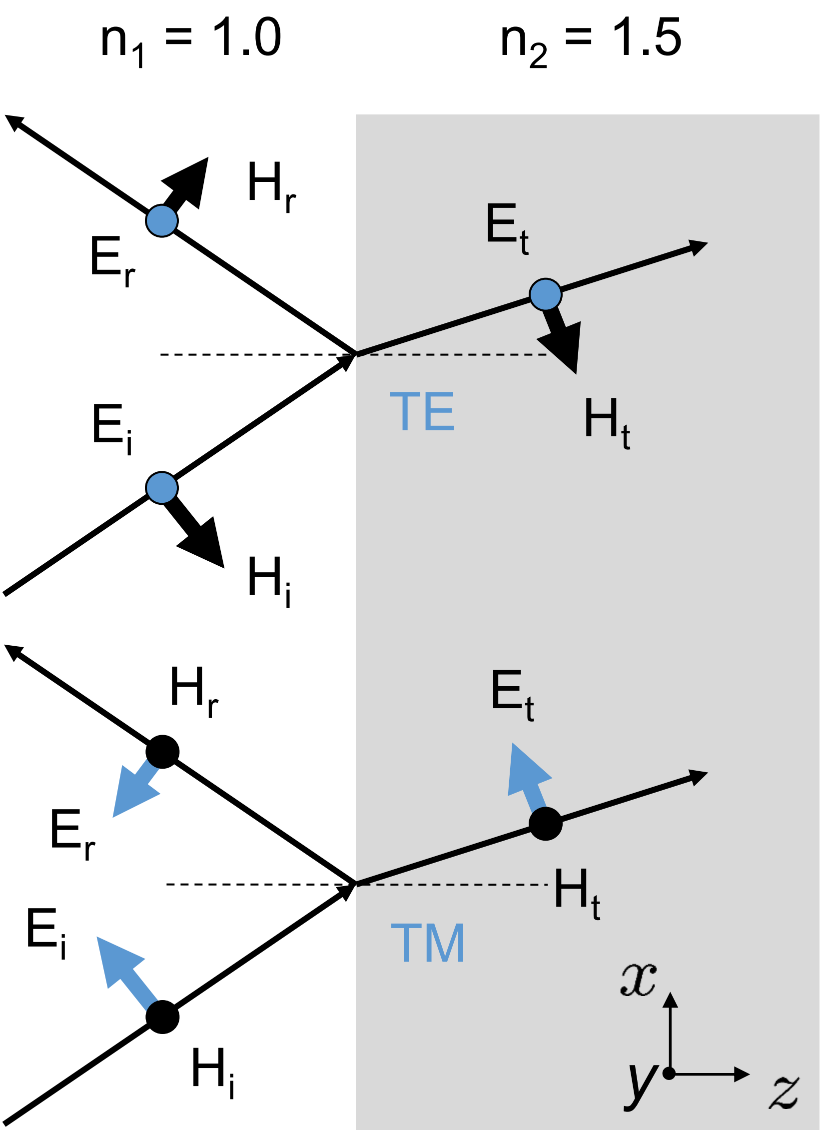
$$t_{TE} = \frac{E_o^t}{E_o^i} = \frac{2\mu_2 k_{iz}}{\mu_2 k_{iz} + \mu_1 k_{tz}}$$

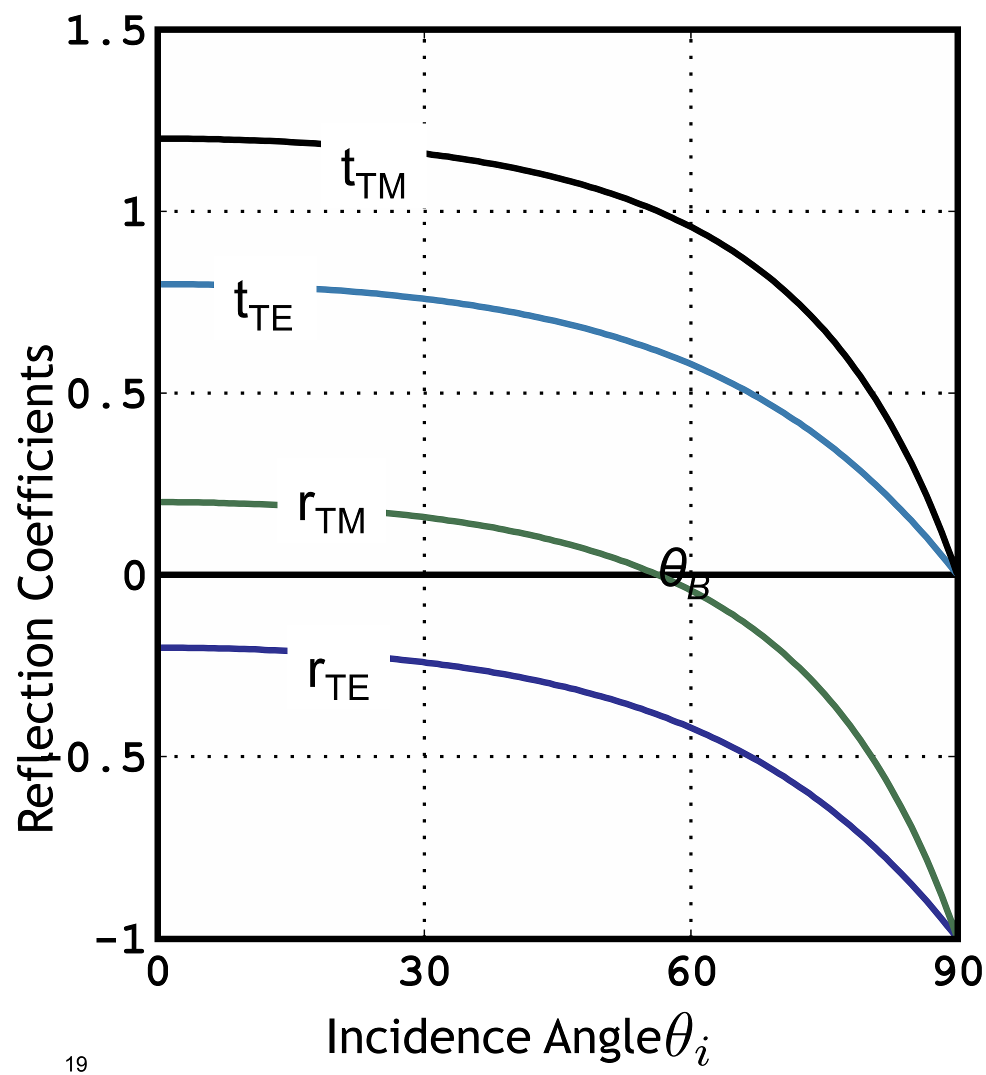
TM Polarization

$$r_{TM} = \frac{E_o^r}{E_o^i} = \frac{\epsilon_2 k_{iz} - \epsilon_1 k_{tz}}{\epsilon_2 k_{iz} + \epsilon_1 k_{tz}}$$

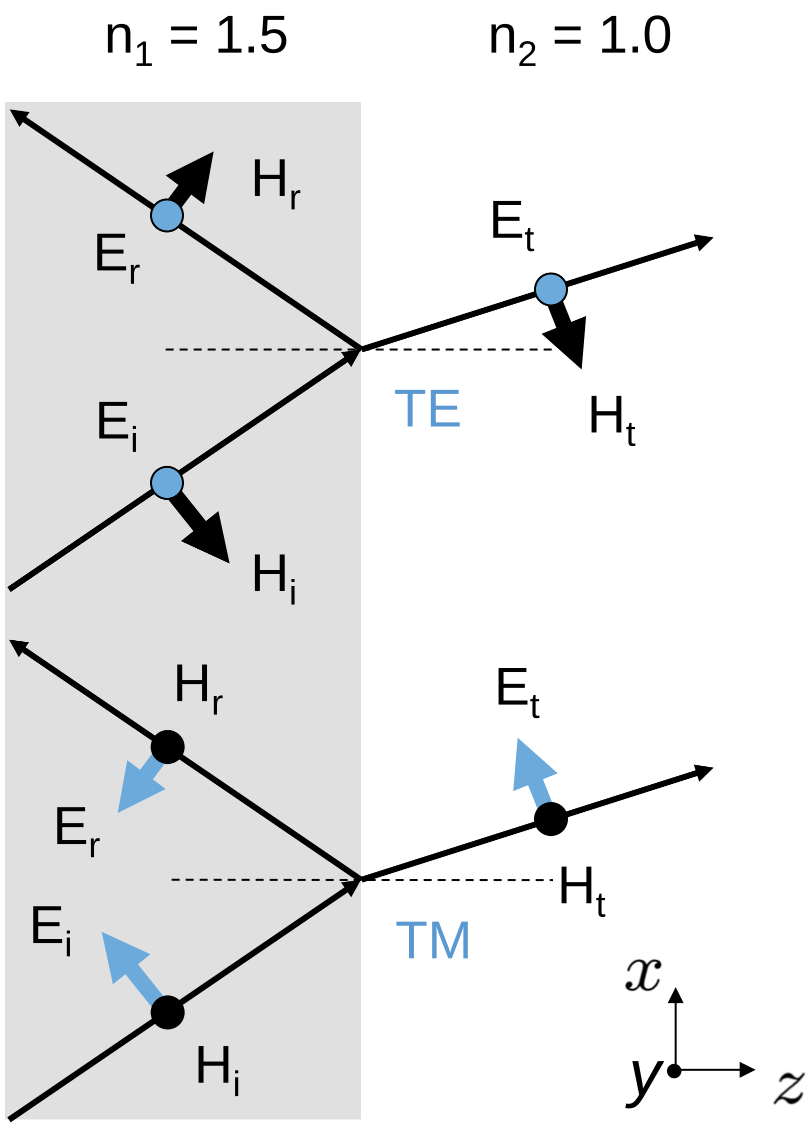
$$t_{TM} = \frac{E_o^t}{E_o^i} = \frac{2\epsilon_2 k_{iz}}{\epsilon_2 k_{iz} + \epsilon_1 k_{tz}}$$

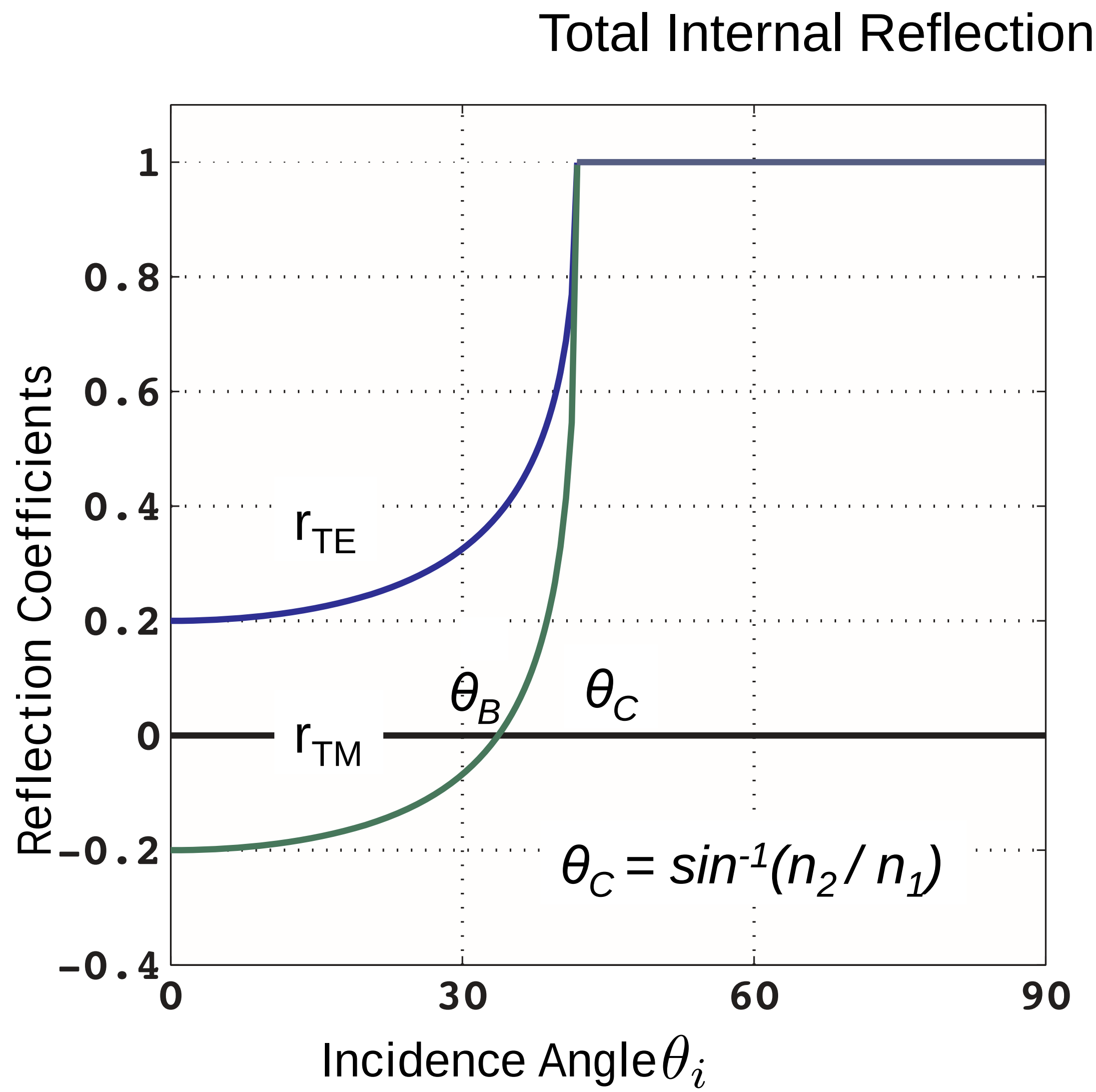
Fresnel Equations





Fresnel Equations

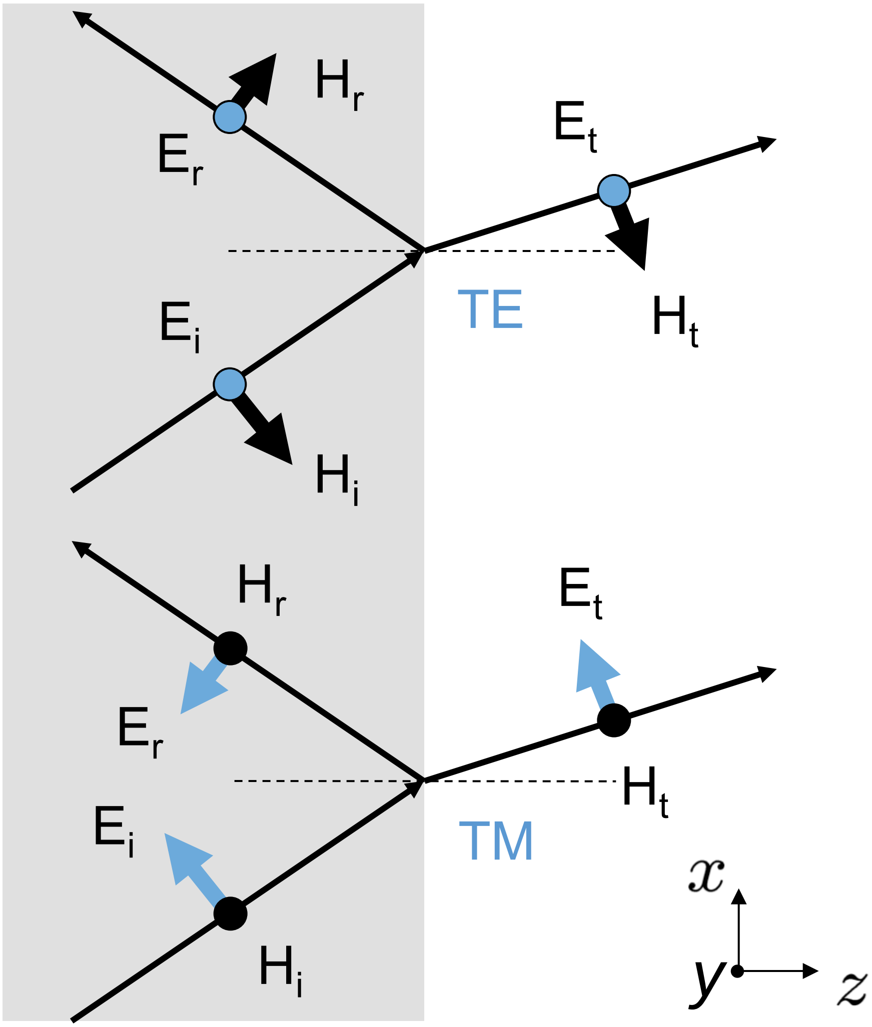


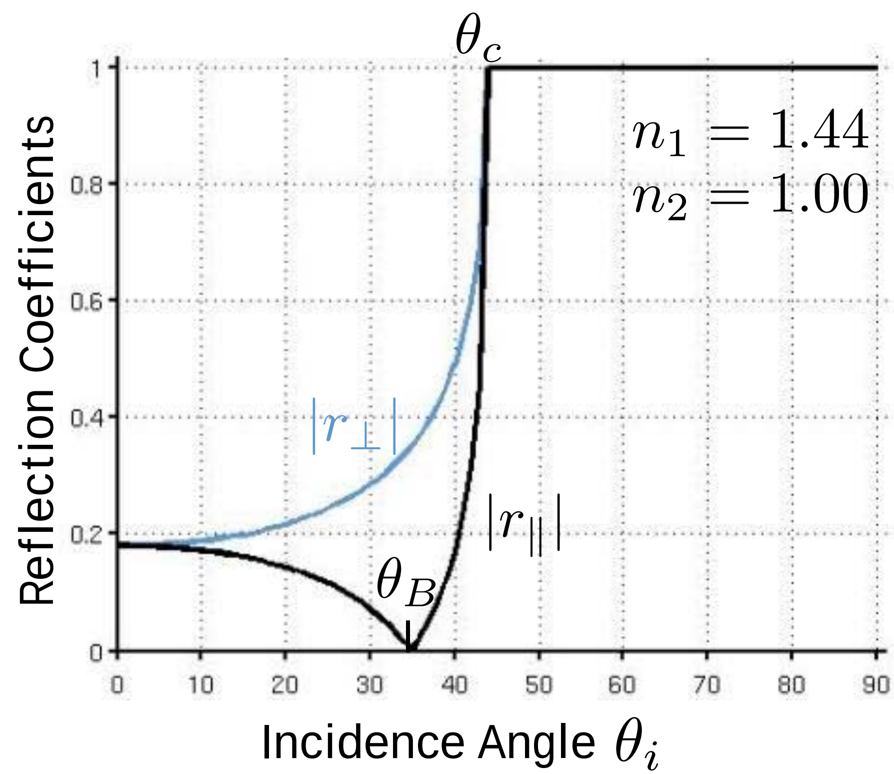


Reflection of Light
(Optics Viewpoint … $\mu_1 = \mu_2$)

TE: $$r_\perp = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

TM: $$r_\parallel = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$





Brewster's Angle
(Optics Viewpoint … $\mu_1 = \mu_2$)

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Sir David Brewster

$r_\perp = 0$ and Snell's Law } Requires $n_1 = n_2$

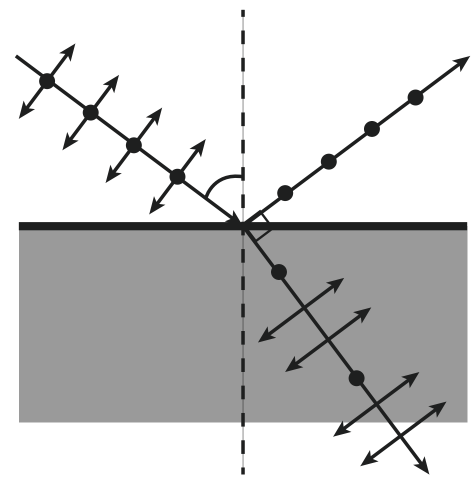
$r_\parallel = 0$ and Snell's Law } Can be Satisfied with

$\theta_B = \arctan(n_2/n_1)$

$\theta_B + \theta_T = 90°$

Unpolarized Incident Ray

TE Polarized Reflected Ray

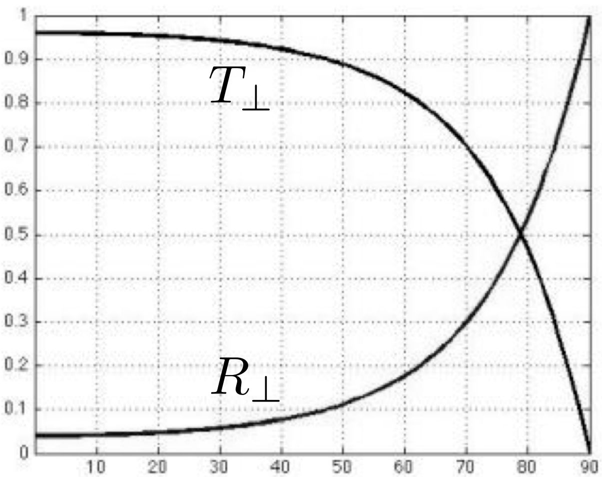


Slightly Polarized Reflracted Ray

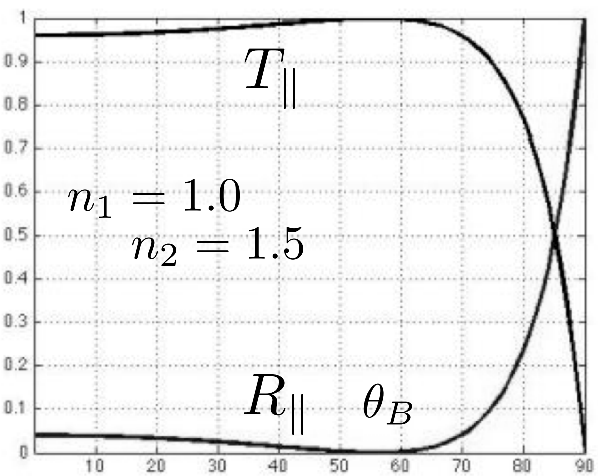
Light incident on the surface is absorbed, and then reradiated by oscillating electric dipoles (Lorentz oscillators) at the interface between the two media. The dipoles that produce the transmitted (refracted) light oscillate in the polarization direction of that light. These same oscillating dipoles also generate the reflected light. However, dipoles do not radiate any energy in the direction along which they oscillate. Consequently, if the direction of the refracted light is perpendicular to the direction in which the light is predicted to be specularly reflected, the dipoles will not create any TM-polarized reflected light.

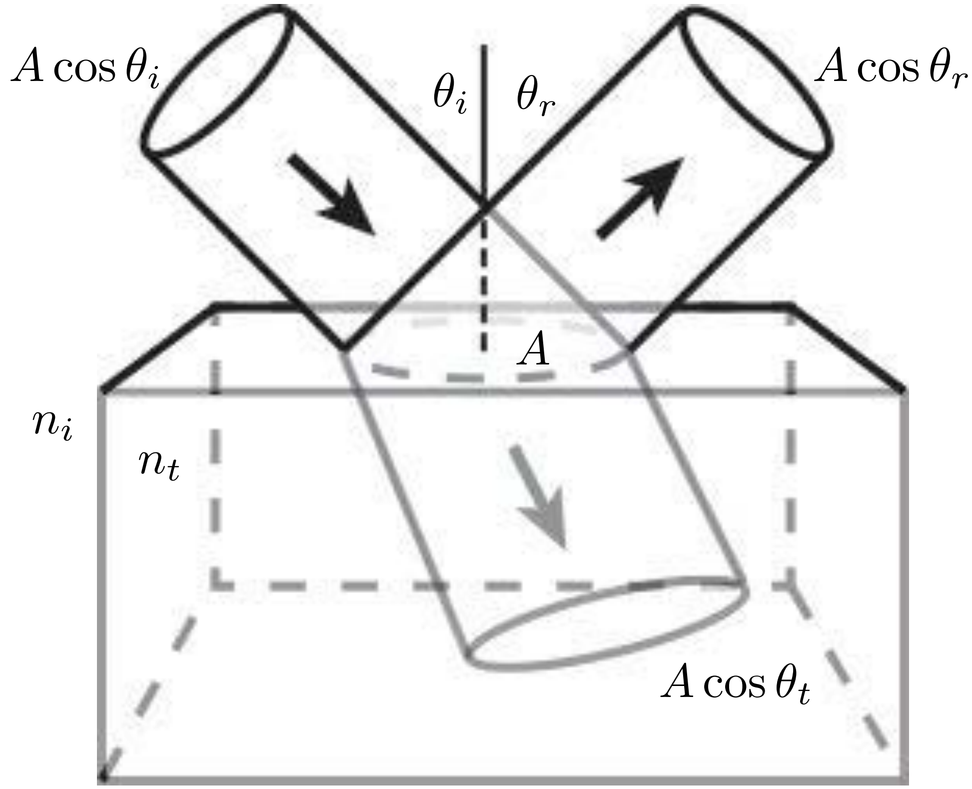
Energy Transport

TE



TM





Cross-sectional Areas

$$\vec{S} = \vec{E} \times \vec{H} \quad \begin{cases} S_i = E_o^i H_o^i \cos\theta_i \\ S_t = E_o^t H_o^t \cos\theta_t \\ S_r = E_o^r H_o^r \cos\theta_r \end{cases}$$

Transmitted Power Fraction:

$$T_s = \frac{S_t}{S_i} = \frac{E_o^t H_o^t \cos\theta_t}{E_o^i H_o^i \cos\theta_i} = \frac{\left(E_o^t\right)^2 \sqrt{\frac{\epsilon_2}{\mu_2}} \cos\theta_t}{\left(E_o^i\right)^2 \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\theta_i} = t_E^2 \frac{N_t}{N_i}$$

$$= \frac{\left(H_o^t\right)^2 \sqrt{\frac{\mu_2}{\epsilon_2}} \cos\theta_t}{\left(H_o^i\right)^2 \sqrt{\frac{\mu_1}{\epsilon_1}} \cos\theta_i} = t_H^2 \frac{M_t}{M_i}$$

$$\frac{\left(E_o^t\right)^2}{\left(E_o^i\right)^2} \equiv t_E^2$$

$$\frac{\left(H_o^t\right)^2}{\left(H_o^i\right)^2} \equiv t_H^2$$

Reflected Power Fraction:

$$R_s = \frac{S_r}{S_i} = \frac{E_o^r H_o^r \cos\theta_r}{E_o^i H_o^i \cos\theta_i} = \frac{\left(E_o^r\right)^2 \sqrt{\frac{\epsilon_1}{\mu_1}}}{\left(E_o^i\right)^2 \sqrt{\frac{\epsilon_1}{\mu_1}}} = r_E^2$$

$$= \frac{\left(H_o^r\right)^2 \sqrt{\frac{\mu_1}{\epsilon_1}}}{\left(H_o^i\right)^2 \sqrt{\frac{\mu_1}{\epsilon_1}}} = r_H^2$$

$$\frac{\left(E_o^r\right)^2}{\left(E_o^i\right)^2} \equiv r_E^2$$

$$\frac{\left(H_o^r\right)^2}{\left(H_o^i\right)^2} \equiv r_H^2$$

… and from ENERGY CONSERVATION we know:

$$T_S + R_S = 1$$

Summary CASE I:

E-field is polarized perpendicular to the plane of incidence

… then E-field is parallel (tangential) to the surface, and continuity of tangential fields requires that:

$$1 + r_E = t_E$$

$$T_S + R_S = 1$$

$$(t_E)^2 \frac{N_t}{N_i} + (r_E)^2 = 1$$

$$(1 + r_E)^2 \frac{N_t}{N_i} + (r_E)^2 = 1$$

$$r_E = \frac{E_o^r}{E_o^i} = \frac{N_i - N_t}{N_i + N_t}$$

$$t_E = 1 + r_E = \frac{E_o^t}{E_o^i} = \frac{2N_i}{N_i + N_t}$$

Summary CASE II:

H-field is polarized perpendicular to the plane of incidence

… then H-field is parallel (tangential) to the surface, and continuity of tangential fields requires that:

$$1 + r_H = t_H$$

$$T_S + R_S = 1$$

$$(t_H)^2 \frac{M_t}{M_i} + (r_H)^2 = 1$$

$$(1 + r_H)^2 \frac{M_t}{M_i} + (r_H)^2 = 1$$

$$r_H = \frac{H_o^r}{H_o^i} = \frac{M_i - M_t}{M_i + M_t}$$

$$t_H = 1 + r_H = \frac{H_o^t}{H_o^i} = \frac{2M_i}{M_i + M_t}$$

6.007 Electromagnetic Energy: From Motors to Lasers
Spring 2011