MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.007 – Electromagnetic Energy: From Motors to Lasers
Spring 2011

Problem Set 8: Electromagnetic Waves at Boundaries
Due Friday, April 15, 2011

MATLAB Tips: Dealing with Imaginary Numbers

- MATLAB treats the variables `i` and `j` as $\sqrt{-1}$ unless you've set them explicitly to something else.
- If you make a complex number `x`, e.g. `x=1+i`, you can find the real and imaginary parts of `x` using the functions `real(x)` and `imag(x)`, respectively.
- To find the magnitude of a complex number in MATLAB, use the function `abs()`. For example, given a complex number `x`, the magnitude $|x|$ is `abs(x)`.
- Likewise, to find the phase of a complex number in MATLAB, use the function `angle()`.
- And one general note, MATLAB is built around vectors, and as you may recall, vector multiplication is not element by element but rather row by column. Here, we're interested in element by element operations since we just want to evaluate an equation (e.g., for the relative dielectric constant given by the Lorentz model equation) at a set of points. So if we have a vector `omega` with all of our frequencies, to square each frequency you use `omega.^2`. The "`.`" means to do each element separately. Similarly, to get the free space wavelength corresponding to those ω, we would do `lambda=2*pi*3*10^8./omega`, which would give us the vector `lambda` with our wavelengths in meters. Usually, if you forget to use the "`.`" version of the operator, you'll get an error in MATLAB.

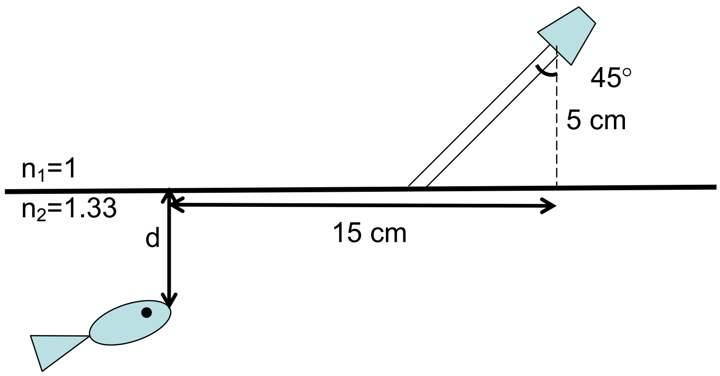
Problem 8.0 – Make Your Own Exam Problem for Midterm II by April 13

Write an original exam problem (with solutions) and turn it in (on a separate sheet of paper from the rest of the problem set) in class by Wednesday, April 13 (ahead of this problem set due date). This problem should be over material covered since the last exam. The most creative and appropriate problem will be selected to be one of the problems for Midterm II, so you have a lot of incentive for writing a great problem!

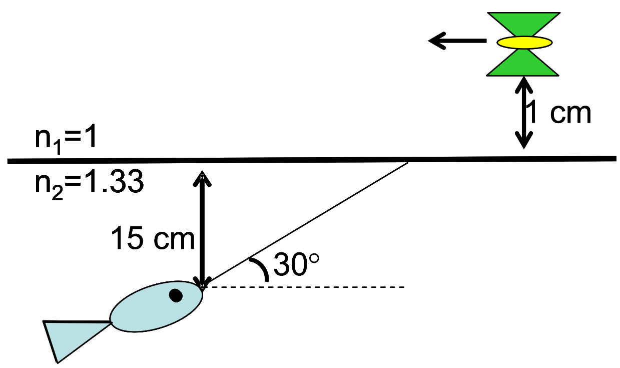
Problem 8.1 – Snell's Law

In ray optics, it is useful to use Snell's Law at an interface between two materials: $n_1 sin(\theta_1) = n_2 sin(\theta_2)$.

(a) Imagine a (collimated) beam of light being shone down to an air-water interface from 5 cm above the water at 45° from the normal. The index of air is taken to be 1 and the index of water is taken to be 1.33. A fish is swimming at 15 cm horizontal distance away from the light as shown. At what depth will the fish see the beam of light?



(b) Now the fish is 15 cm deep, looking up at the water at 30° as shown and there is a fly skimming the water surface 1 cm above the water and changing his position. Can the fish see the fly? Please explain your answer.



Problem 8.2 – Frustrated Total Internal Reflection

This problem explores the phenomenon of frustrated total internal reflection and the more general math that goes with it.

In lecture, we discussed what happens when total internal reflection is frustrated by bringing a second medium (e.g., glass) into the evanescent field of the reflected wave. Figure 1 shows a schematic of the physical setup of frustrated internal reflection, where light which would be reflected internally inside a glass waveguide is able to transmit across an air gap into another piece of glass.

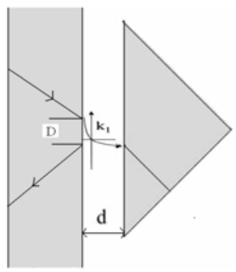


Figure 1: Schematic of frustrated internal reflection.

To simplify our modeling of the above system, we'll look only in the direction across the air gap, assuming that the incoming angle is such that total internal reflection occurs inside the first piece of glass, and therefore, the field in the air gap is evanescent. Figure 2 shows a schematic in 1D of the glass-air-glass transition.

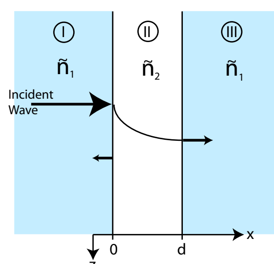


Figure 2: 1D schematic of coupling via evanescent field.

We set up the following equations for the electric field in the various regions in Figure 2, setting the incident wave's magnitude to 1 so that the reflection and transmission coefficients (r_n and t_n) can be solved for directly:

$$E_{y,I}(x,t) = e^{j(\omega t - k_{1x}x)} + r_n e^{j(\omega t + k_{1x}x)}$$
$$E_{y,II}(x,t) = Ae^{-\alpha x}e^{j\omega t} + Be^{\alpha x}e^{j\omega t}$$
$$E_{y,III}(x,t) = t_n e^{j(\omega t - k_{1x}x)}$$

Assume that $\mu = \mu_0$ in all three regions.

(a) Using the boundary condition on the tangential electric field at $x = 0$ and $x = d$, find two equations relating the unknowns r_n, A, B, and t_n.

(b) Find the tangential magnetic fields using Faraday's Law:

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

Then, using the boundary condition on the tangential magnetic field at $x = 0$ and $x = d$, find two equations relating the unknowns r_n, A, B, and t_n.

(c) Now that you have four equations and four unknowns, set up a matrix equation in the form of $\mathbf{M}\vec{x} = \vec{C}$, where:

$$\vec{x} = \begin{pmatrix} r_n \\ A \\ B \\ t_n \end{pmatrix}$$

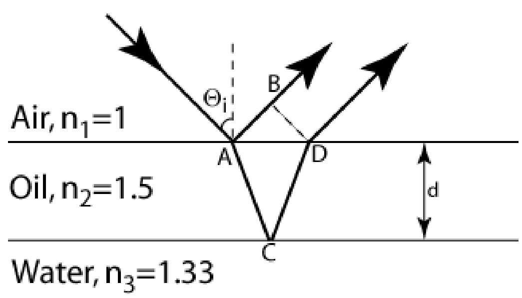
With this matrix equation, you could solve for the complex amplitudes r_n, A, B, and t_n. We'll see this again when we cover tunneling in quantum mechanics.

(d) Using MATLAB, find the transmitted intensity, $|t_n|^2$, if the incident angle in region I (glass) is $\theta = 45°$, the index of glass is $n_1 = 1.5$, the index of air is $n_2 = 1$, the free space wavelength is $\lambda_0 = 640$ nm, and the air gap is $d = 100$ nm.

You'll need to numerically solve the system of equations represented by your matrix in (c) using MATLAB. To find the inverse of **M**, use the MATLAB function `inv()` (e.g., `x=inv(M)*C`).

Problem 8.3 – Thin Film Interference

In class we have seen the Fresnel equations for reflected and transmitted wave amplitudes. These equations assume that the materials are semi-infinite (that they continue for ever). If we look at reflections from sections of materials with finite thickness we have to take into account interference phenomenon in addition to the transmission/reflection amplitudes for semi-infinite material boundaries. In this problem we will look at the reflection of light from a film of oil on top of water. See the diagram below.



Geometry for Problem 8.4, not drawn to scale.

There can be a peak in reflected intensity only if the difference in phase gained between path AB and path ACD is equal to some integer multiple of 2π. $|\phi_{AB} - \phi_{ACD}| = 2\pi N$.

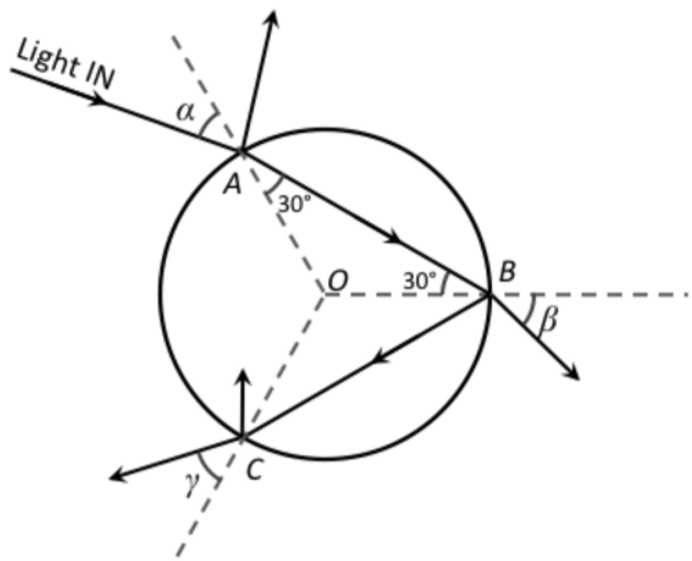
(a) Write an expression for the phase difference in terms of Θ_i, n_1, n_2, d, and λ_0.

(b) If the oil film is 1 μm thick, at what angles do we see the first strong reflections for $\lambda = 450nm$, $\lambda = 530nm$, and $\lambda = 630nm$?

(c) Using the results from above, why do we see a rainbow of colors on an oil slick?

(d) For TM polarized light, what happens when the angle for constructive interference is equal to Brewster's Angle for the air-oil interface?

Problem 8.4 – Rainbows



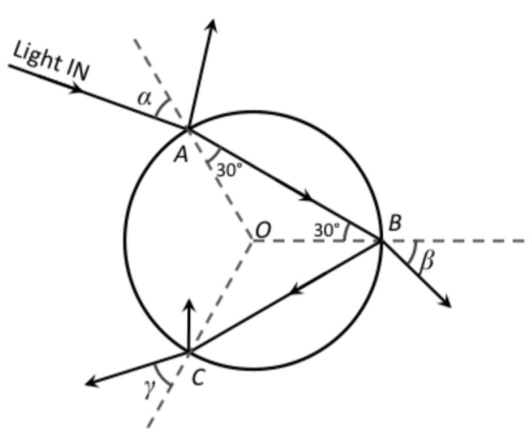
A very narrow beam of unpolarized red light of intensity I_o is incident (at A) on a spherical water drop (see figure above). At A, some of the light is reflected and some enters the water drop. The refracted light reaches the surface of the drop at B where some of the light is reflected back into the water, and some emerges into the air. The light that is reflected back into the water reaches the surface of the drop at C where some of the light is reflected back into the drop, and some emerges into the air. The index of refraction of water for the red light is n_{red} =1.331.

(a) Using the data form the figure, what is the angle α? (You can leave this answer in the form of an expression.)

(b) What is the intensity of light that refracts into the drop at A? (Take into account both TE and TM polarized light.)

(c) For what value of angle would you find that the light reflected at A is entirely TE polarized ? (You can leave this answer in the form of an expression.)

(d) What is the dominant polarization of light that emerges at B? Explain.



(e) For blue light, the index of refraction in water is 1.343. The speed of blue light in water is therefore about 1% slower than that of red light. In the figure above, showing the red-light path, assume that the incoming narrow beam of light also contains blue-light and draw the trajectory that the blue-light beam would take after it enters the water droplet at A.

(f) If you look at the rainbow in the sky you will notice that the blue band of color is closest to the ground and the red color band is highest up. Using the figure below explain why the colors are ordered in this manner.