Second-Order Systems

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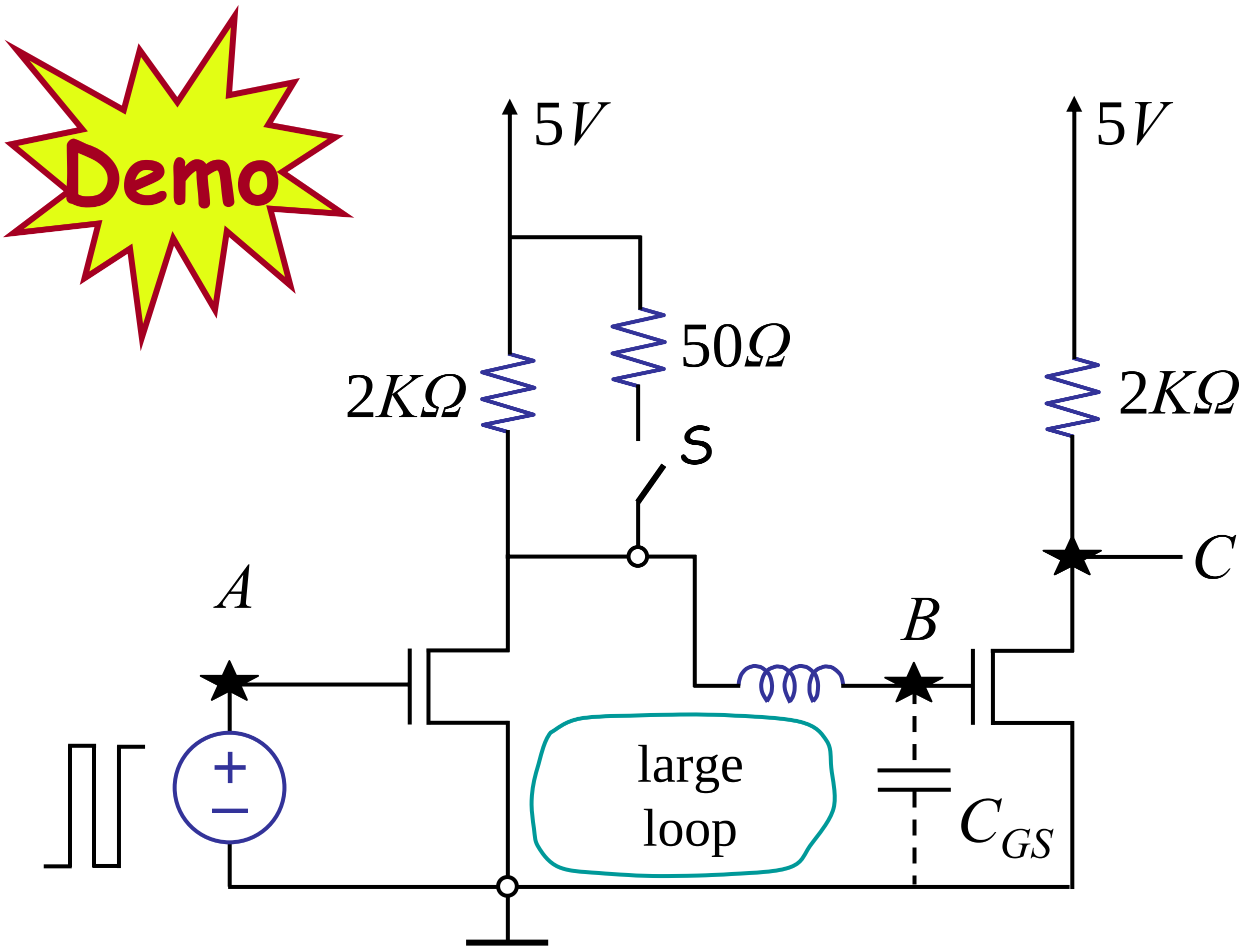
Second-Order Systems



Our old friend, the inverter, driving another. The parasitic inductance of the wire and the gate-to-source capacitance of the MOSFET are shown

[Review complex algebra appendix for next class]

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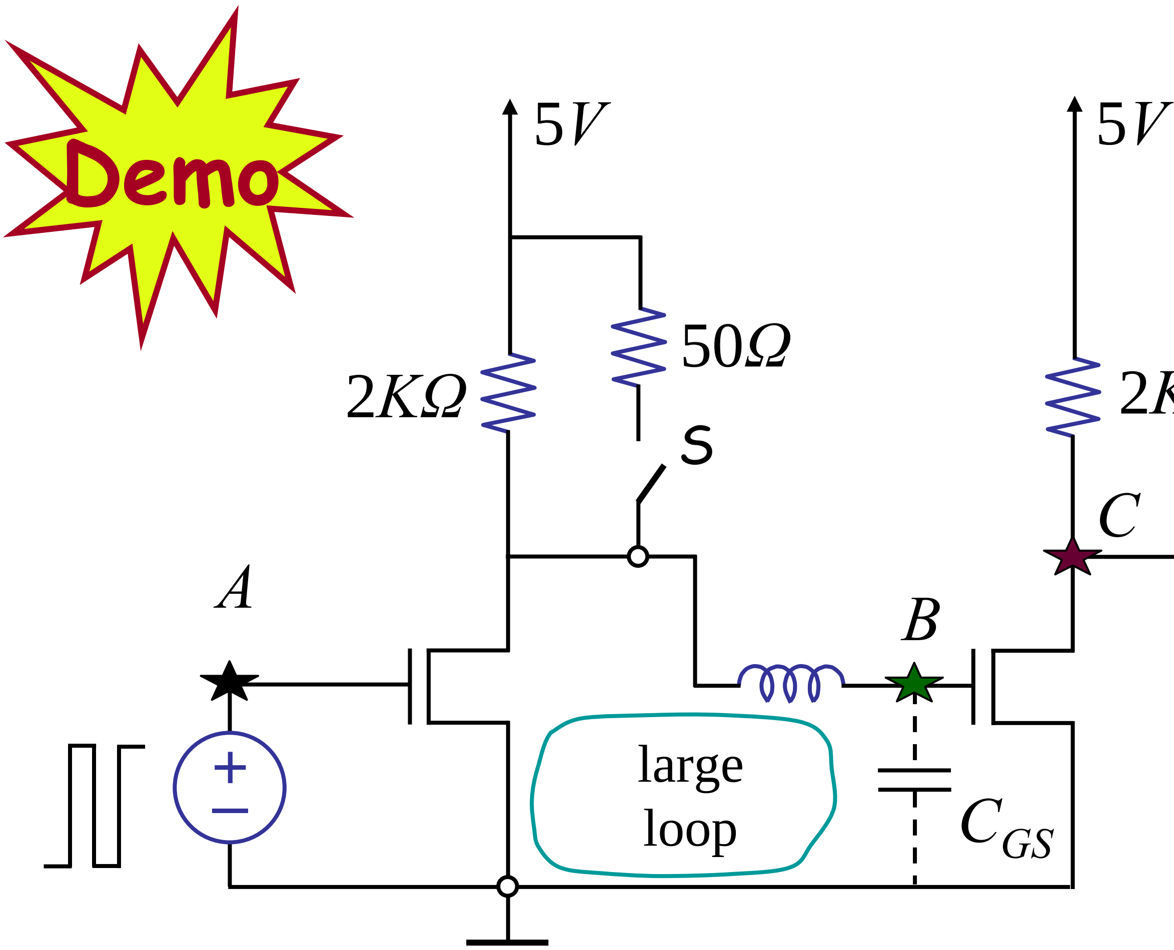
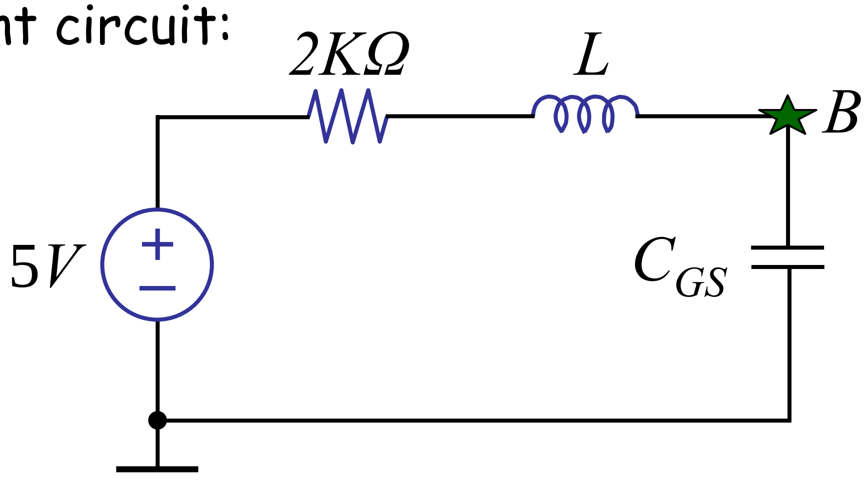
Second-Order Systems



Relevant circuit:



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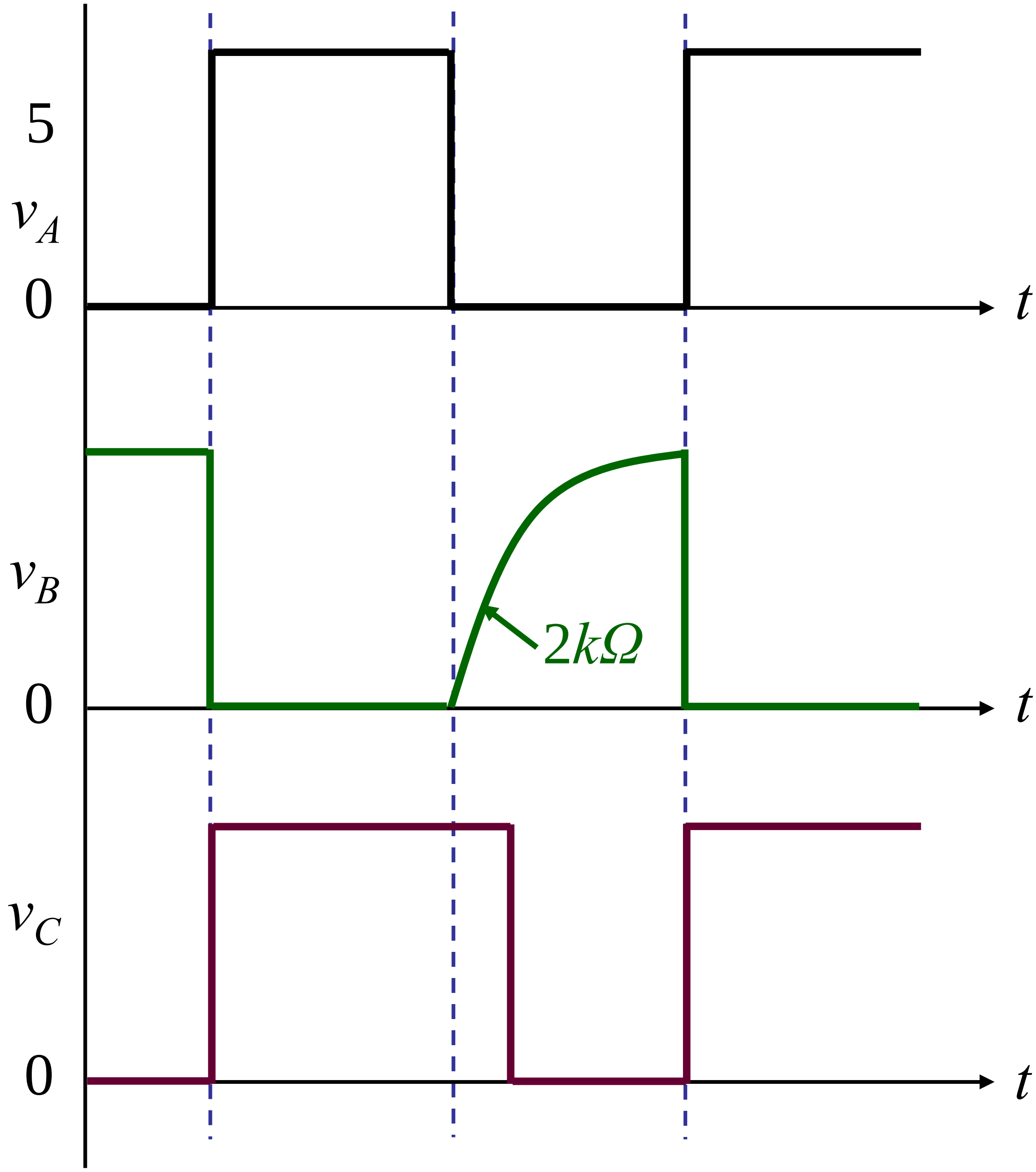
Observed Output $2k\Omega$



Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance

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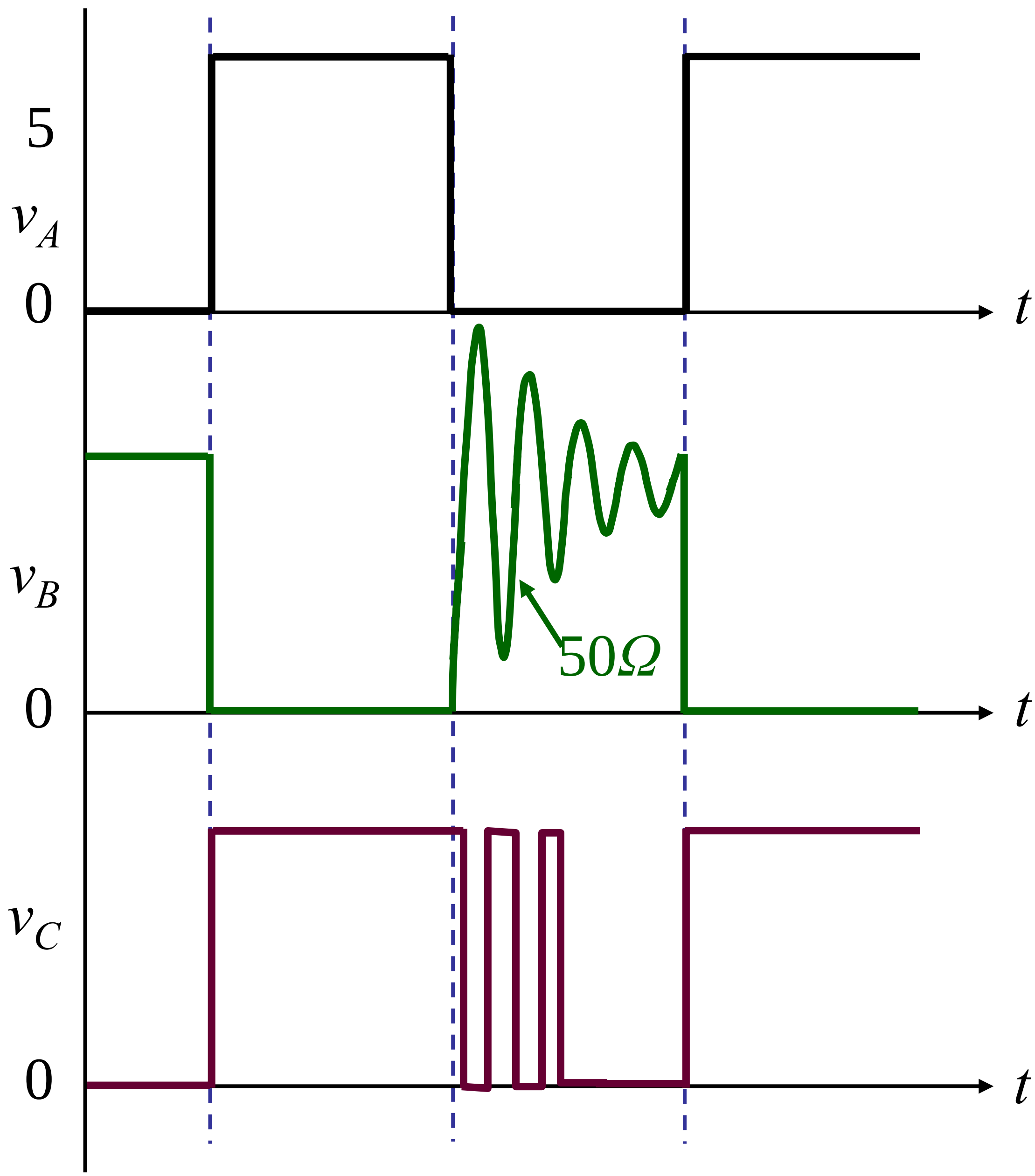
Observed Output ~50Ω



Huh!

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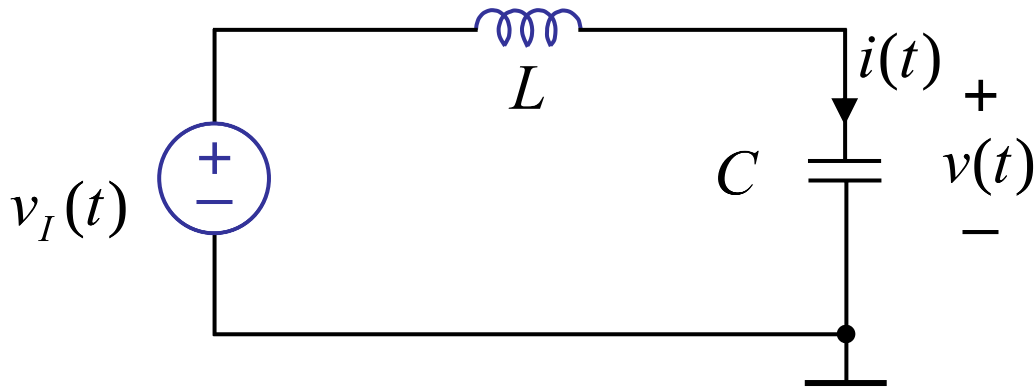
First, let's analyze the LC network



Node method:

$$i(t) = C\frac{dv}{dt}$$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = C\frac{dv}{dt}$$

$$\frac{1}{L}(v_I - v) \qquad = C\frac{d^2v}{dt^2}$$

$$\underbrace{LC}_{\text{time}^2}\frac{d^2v}{dt^2} + v = v_I$$

Recall

$$v_I - v = L\frac{di}{dt}$$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = i$$

v, i state variables

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Solving

Recall, the method of homogeneous and particular solutions:

1. Find the particular solution.
2. Find the homogeneous solution.
   ↓
   4 steps
3. The total solution is the sum of the particular and homogeneous.

   Use initial conditions to solve for the remaining constants.

$$v = v_P(t) + v_H(t)$$

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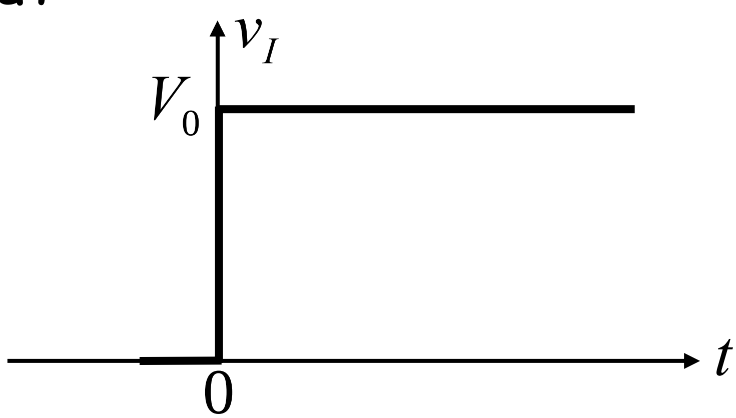
Let's solve

$$LC\frac{d^2v}{dt^2}+v=v_I$$

For input



And for initial conditions

$v(0) = 0 \qquad i(0) = 0 \qquad$ [ZSR]

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① Particular solution

$$LC\frac{d^2 v_P}{dt^2} + v_P = V_0$$

$$v_P = V_0$$ is a solution.

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2 Homogeneous solution

Solution to

$$LC\frac{d^2 v_H}{dt^2} + v_H = 0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

A Assume solution of the form*

$$v_H = Ae^{st} \quad , \quad A,\ s\ =\ ?$$

so, $$LC\cancel{A}s^2\cancel{e^{st}} + \cancel{Ae^{st}} = 0$$

B $$s^2 = -\frac{1}{LC}$$ characteristic equation

$$s = \pm j\sqrt{\frac{1}{LC}}$$

$$j = \sqrt{-1}$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

C Roots $$s = \pm j\omega_o$$

General solution,

D $$v_H = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

*Differential equations are commonly solved by guessing solutions

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③ Total solution

$$v(t) = v_P(t) + v_H(t)$$

$$v(t) = V_0 + A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

Find unknowns from initial conditions.

$$v(0) = 0$$

$$0 = V_0 + A_1 + A_2$$

$$i(0) = 0$$

$$i(t) = C\frac{dv}{dt}$$

$$i(t) = CA_1 j\omega_o e^{j\omega_o t} - CA_2 j\omega_o e^{-j\omega_o t}$$

so, $$0 = CA_1 j\omega_o - CA_2 j\omega_o$$

or, $$A_1 = A_2$$

$$-V_0 = 2A$$

$$A_1 = -\frac{V_0}{2}$$

so, $$v(t) = V_0 - \frac{V_0}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)$$

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③ Total solution

Remember Euler relation

$$e^{jx} = \cos x + j \sin x$$

(verify using Taylor's expansion)

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x$$

so,

$$v(t) = V_0 - V_0 \cos \omega_o t$$

$$i(t) = CV_0 \omega_o \sin \omega_o t$$

where

$$\omega_o = \frac{1}{\sqrt{LC}}$$

The output looks sinusoidal

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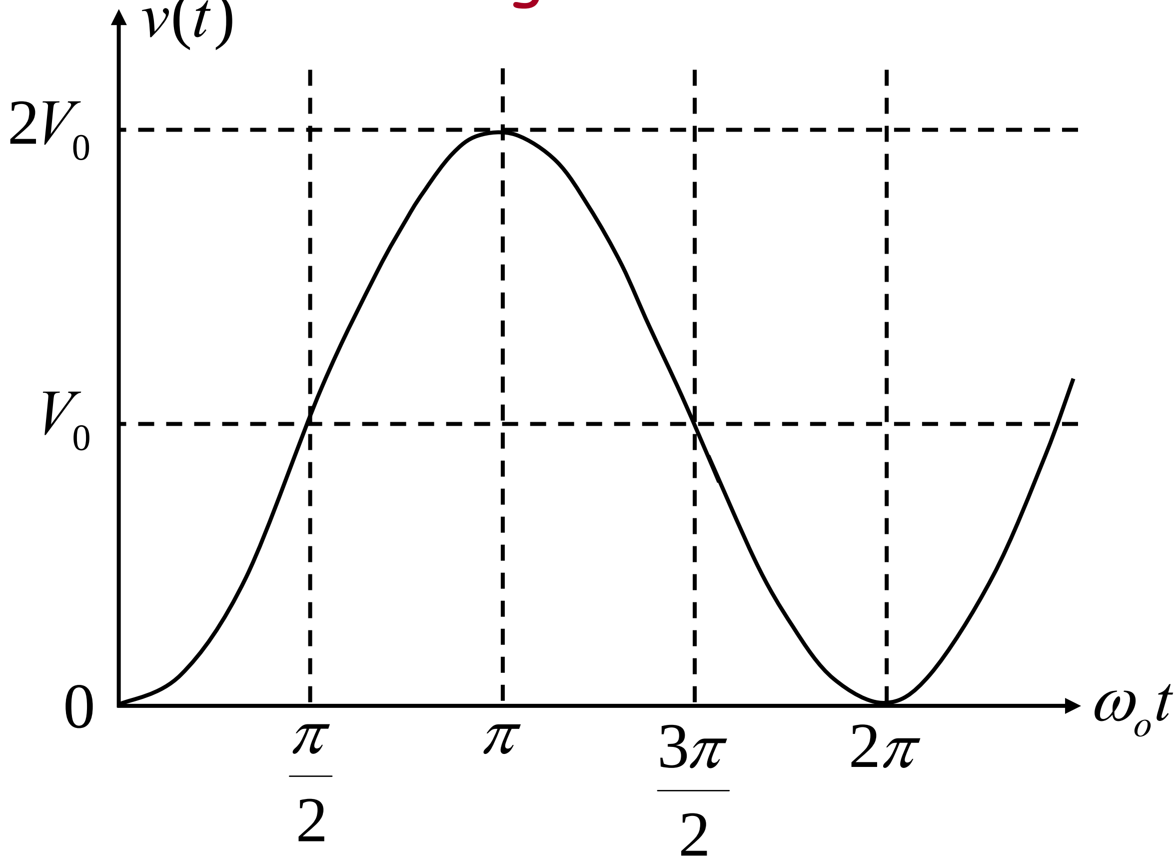
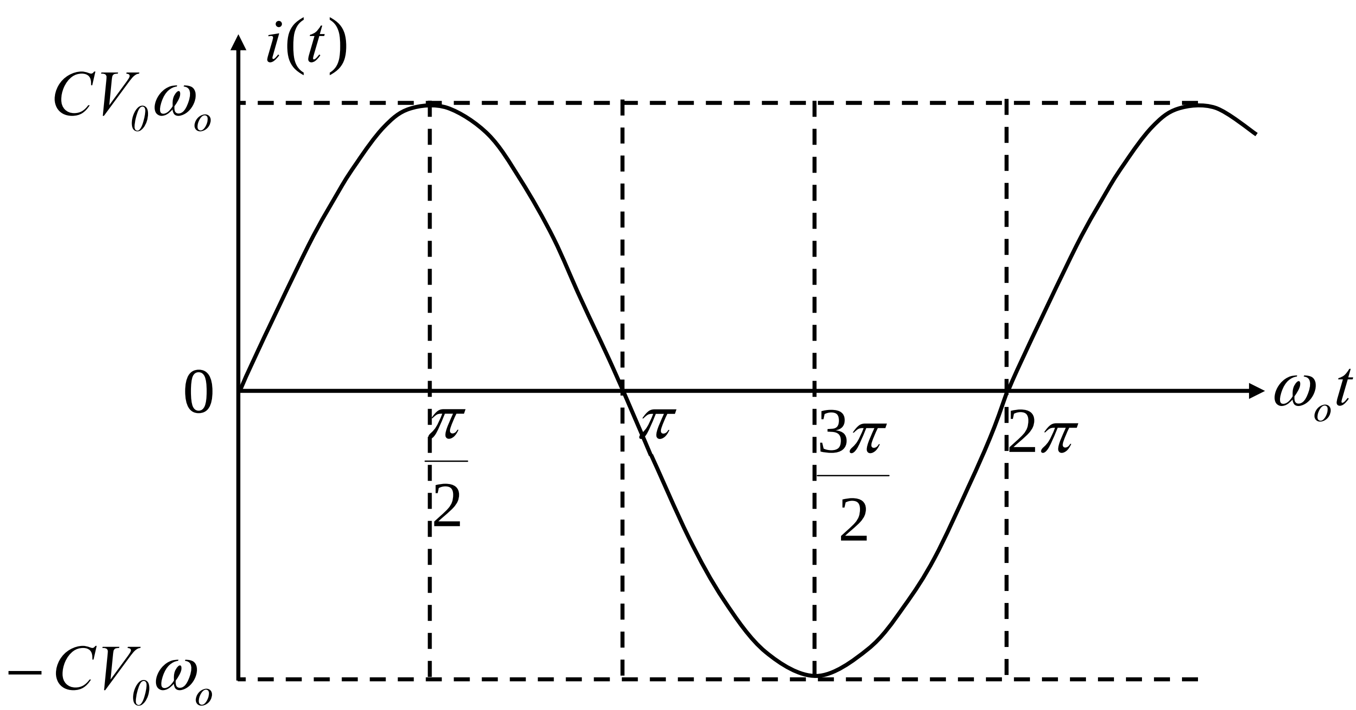
Plotting the Total Solution





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Summary of Method

1. Write DE for circuit by applying node method.

2. Find particular solution v_P by guessing and trial & error.

3. Find homogeneous solution v_H

   A. Assume solution of the form Ae^{st}.

   B. Obtain characteristic equation.

   C. Solve characteristic equation for roots s_i.

   D. Form v_H by summing $A_i e^{s_i t}$ terms.

4. Total solution is $v_P + v_H$, solve for remaining constants using initial conditions.

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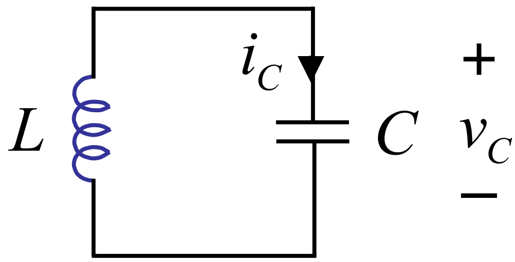
Example

What if we have:



$v_C(0) = V$

$i_C(0) = 0$

We can obtain the answer directly from the homogeneous solution $(V_0 = 0)$.

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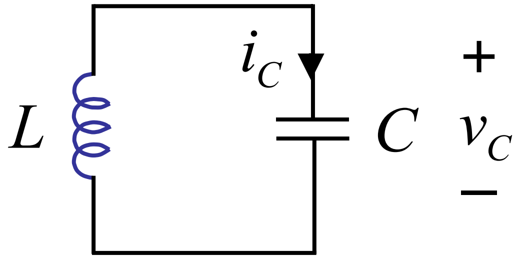
Example



$v_C(0) = V$

$i_C(0) = 0$

We can obtain the answer directly from the homogeneous solution $(V_0 = 0)$.

$$v_C(t) = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

$v_C(0) = V$

$$V = A_1 + A_2$$

$i_C(0) = 0$

$$0 = CA_1 j\omega_o - CA_2 j\omega_o$$

or $\quad A_1 = A_2 = \dfrac{V}{2}$

or $\quad v_C = \dfrac{V}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)$

$$v_C = V \cos \omega_o t$$

$$i_C = -CV\,\omega_o \sin \omega_o t$$

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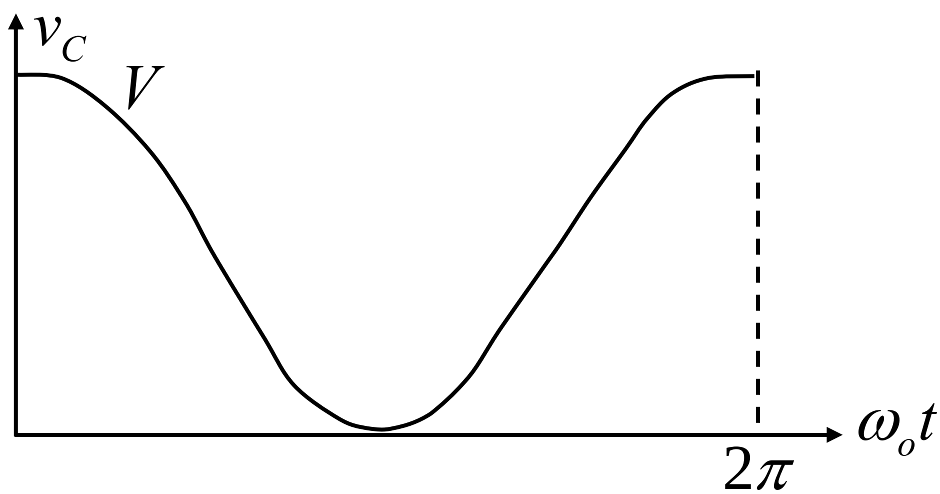
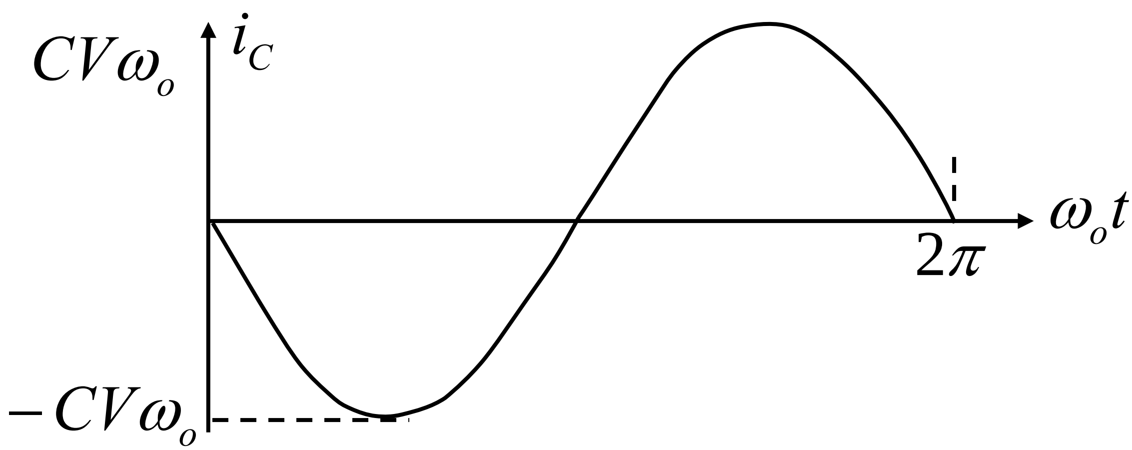
Example





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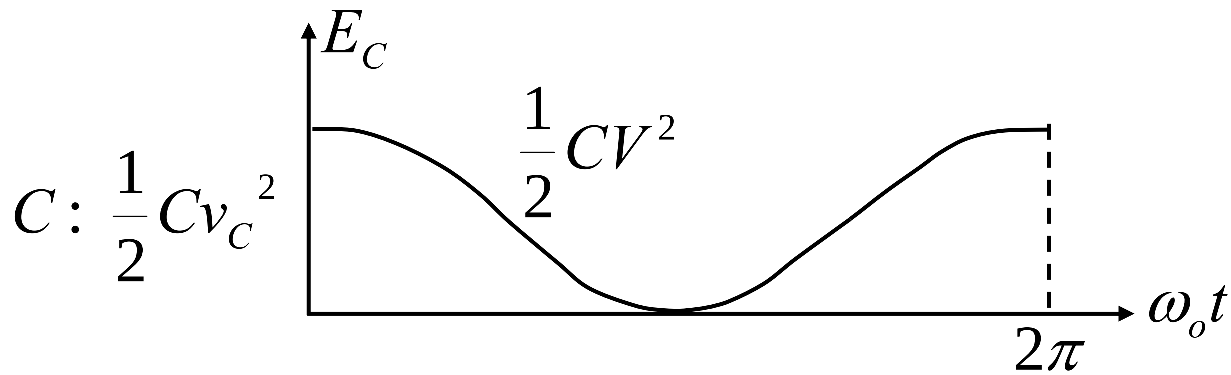
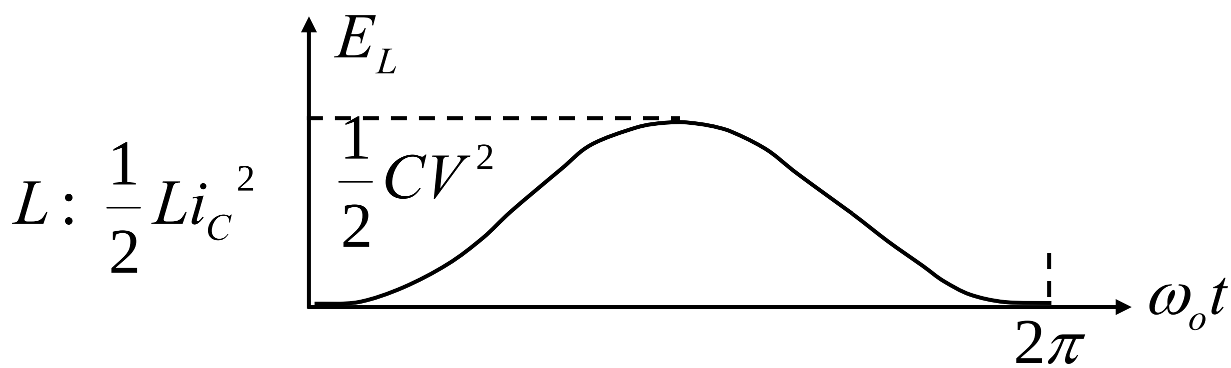
Energy





Notice $\quad \frac{1}{2}Cv_C{}^2 + \frac{1}{2}Li_C{}^2 = \frac{1}{2}CV^2$

Total energy in the system is a constant, but it sloshes back and forth between the Capacitor and the inductor

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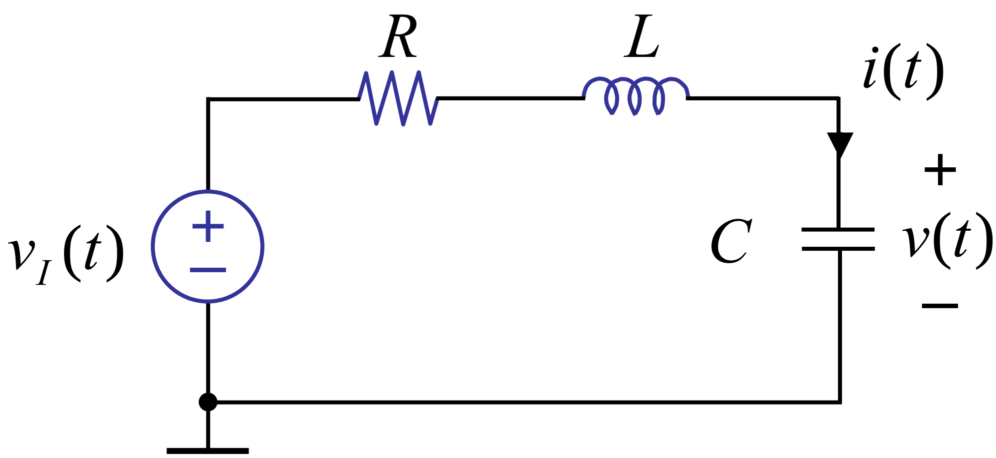
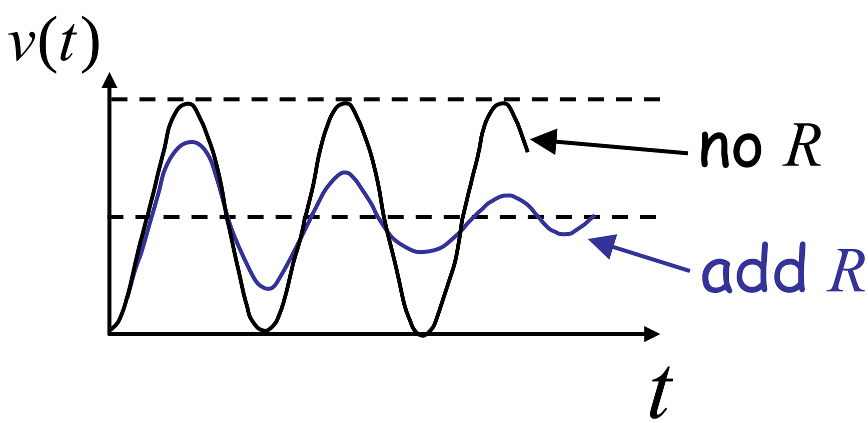
RLC Circuits





Damped sinusoids with R – remember demo!

See A&L Section 12.2

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