Solutions toProblem Set # 4
Macro 453

October 18, 2003

1 Problem 1: Incomplete Markets and Asset Prices

This problem investigates the effects of market incompleteness on asset pricing following Constantinides and Duffie (1996). There is a continuum of individuals with identical CRRA preferences: $u(c) = c^{1-\gamma}/(1-\gamma)$ and subjective discount factor is β.

1.1 Part (a)

Write down the Euler equation for individual i and asset j. This equation should relate the gross return R^j_{t+1} to the growth rate of consumption for individual i: c^i_{t+1}/c^i_t.

The standard Euler equation for each individual i and asset j will be

$$1 = \beta E_t \left[R^j_{t+1} \frac{u'\left(c^i_{t+1}\right)}{u'\left(c^i_t\right)} \right]$$

and with our preferences specification

$$1 = \beta E_t \left[R^j_{t+1} \left(\frac{c^i_{t+1}}{c^i_t} \right)^{-\gamma} \right]$$

1.2 Part (b)

Now assume that the growth rate of individual consumption satisfies,

$$\frac{c_{t+1}^i}{c_t^i} = \frac{c_{t+1}}{c_t}\varepsilon_{t+1}^i, \tag{1}$$

where c_{t+1}/c_t is the growth rate of aggregate consumption. Here ε_{t+1}^i represents the idiosyncratic component of consumption growth. Conditional on σ_{t+1}^2, c_{t+1}/c_t and R_{t+1}^j, assume ε_{t+1}^i is independent across individuals and $\log \varepsilon_{t+1}^i$ is distributed $N(-\frac{1}{2}\sigma_{t+1}^2, \sigma_{t+1}^2)$. The cross sectional variance parameter, σ_{t+1}^2, is a random variable from the point of view of time t, it becomes known at time $t+1$. Note that σ_{t+1}^2 is not assumed to be independent of c_{t+1}/c_t and R_{t+1}^j (i.e. the three variables may be correlated with each other).

Use the individual Euler equation from (a) to show that:

$$1 = \beta E_t \left[\Phi\left(\sigma_{t+1}^2\right) \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}^j \right] \tag{2}$$

where

$$\Phi\left(\sigma_{t+1}^2\right) \equiv E\left[\left(\varepsilon_{t+1}^i\right)^{-\gamma} | \sigma_{t+1}^2\right] = \exp\left\{\frac{\gamma\left(1+\gamma\right)}{2}\sigma_{t+1}^2\right\}.$$

From part (a)

$$\begin{aligned} 1 &= \beta E_t \left[R_{t+1}^j \left(\frac{c_{t+1}^i}{c_t^i}\right)^{-\gamma} \right] = \\ &= \beta E_t \left[R_{t+1}^j \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(\varepsilon_{t+1}^i\right)^{-\gamma} \right] \end{aligned}$$

Recall that we know $\left(\log \varepsilon_{t+1}^i \mid X\right) \sim N(-\frac{1}{2}\sigma_{t+1}^2, \sigma_{t+1}^2)$ where

$$X' = \begin{pmatrix} \sigma_{t+1}^2 & \frac{c_{t+1}}{c_t} & R_{t+1}^j \end{pmatrix}$$

We can rewrite

$$1 = \beta E_t \left[R_{t+1}^j \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \exp\left\{\log\left[\left(\varepsilon_{t+1}^i\right)^{-\gamma}\right]\right\} \right]$$

and using the law of iterated expectations

$$\begin{aligned} 1 &= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} E_t\left[\exp\left\{\log\left[\left(\varepsilon^i_{t+1}\right)^{-\gamma}\right]\right\} \mid X\right]\right] = \\ &= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} E_t\left[\exp\left\{-\gamma \log \varepsilon^i_{t+1}\right\} \mid X\right]\right] \end{aligned}$$

From the known properties of the normal distribution we know that if $z \sim N(\mu, \sigma^2)$ and c a constant, then

$$E\left[\exp cz\right] = \exp\left(c\mu + \frac{1}{2}c^2\sigma^2\right)$$

It follows

$$\begin{aligned} 1 &= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \exp\left\{\frac{1}{2}\gamma\sigma^2_{t+1} + \frac{1}{2}\gamma^2\sigma^2_{t+1}\right\}\right] = \\ &= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \exp\left\{\frac{\gamma\left(1+\gamma\right)}{2}\sigma^2_{t+1}\right\}\right] \end{aligned}$$

which is what we wanted to prove!

1.3 Part (c)

Specialize the above by assuming that,

$$\sigma^2_{t+1} = A - B\log\left(\frac{c_{t+1}}{c_t}\right).$$

Show that,

$$1 = \hat{\beta} E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\hat{\gamma}} R^j_{t+1}\right] \tag{3}$$

holds with,

$$\begin{aligned} \hat{\gamma} &= \gamma + \frac{1}{2}\gamma\left(1+\gamma\right)B \\ \hat{\beta} &= \beta\exp\left\{\frac{1}{2}\gamma\left(1+\gamma\right)A\right\} \end{aligned}$$

From part (b) we have

$$\begin{aligned}
1 &= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left\{\frac{\gamma(1+\gamma)}{2}\left[A - B\log\left(\frac{c_{t+1}}{c_t}\right)\right]\right\}\right] = \\
&= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\exp\left\{\frac{\gamma(1+\gamma)}{2}A\right\}\exp\left\{-\frac{\gamma(1+\gamma)}{2}B\log\left(\frac{c_{t+1}}{c_t}\right)\right\}\right] = \\
&= \beta E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\frac{\gamma(1+\gamma)}{2}B}\exp\left\{\frac{\gamma(1+\gamma)}{2}A\right\}\right] = \\
&= \beta\exp\left\{\frac{\gamma(1+\gamma)}{2}A\right\}E_t\left[R^j_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\left[\gamma+\frac{\gamma(1+\gamma)}{2}B\right]}\right]
\end{aligned}$$

which is what we wanted to prove (note we are assuming that A and B are constants).

1.4 Part (d)

How can these results help explain the equity premium puzzle?

Recall that the equity premium puzzle is a quantitative puzzle, not a qualitative one. From the Euler equation (3) for a risky asset j

$$1 = \hat{\beta}E_t\left[R^j_{t+1}\right]E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\hat{\gamma}}\right] + \hat{\beta}Cov\left[R^j_{t+1},\left(\frac{c_{t+1}}{c_t}\right)^{-\hat{\gamma}}\right] \tag{4}$$

and for a risk free rate asset

$$1 = \hat{\beta}R^{RF}_{t+1}E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\hat{\gamma}}\right] \tag{5}$$

Note that what we need to explain the equity premium is a negative covariance term in equation (4) which is consistent with the data: in booms both stock returns and consumption increase and viceversa in recessions. The issue turns out to be quantitative in the sense that to match the equity premium that emerges form the data we need a γ much higher than how it seems to be from empirical studies. This comes from the fact that the covariance term is bounded by the variance of consumption. The standard model implies that what matters is the variance of aggregate consumption, but if we introduce idiosyncratic shocks as in this exercise we can take into account the fact that consumption is much more volatile. Note that from part (c) we can express our Euler Equation as observationally equivalent to the standard one, where the new parameters are $\hat{\gamma}$ and $\hat{\beta}$. And note that for $B > 0$ and $A > 0$ we have that $\gamma < \hat{\gamma}$ and $\beta < \hat{\beta}$, which both make the premium larger! In particular, the usual Mehra and Prescott calculation now implies that what is required to be very high to match the data is now $\hat{\gamma}$ which allows for some B to have a much slower γ since

$$\hat{\gamma} = \gamma + \frac{1}{2}\gamma\left(1+\gamma\right)B$$

2 Problems 2

There is an handout for this in my mail box in front of the office.

3 Problem 3

3.1 Part (a)

Let us define the expected profits of the firm if it undertakes the investment in period 1 $\Pi\left(1\right)$ and the expected profits of the firm if it does not undertake the investment $\Pi\left(0\right)$. Then we have

$$\Pi\left(1\right) = \pi_1 + E\left(\pi_2\right) - I$$

and

$$\Pi\left(0\right) = 0$$

It is then clear that the firm will prefer to undertake investment at period 1 than not to undertake it iff $\Pi\left(1\right) > \Pi\left(0\right)$, *i.e.*

$$\pi_1 + E\left(\pi_2\right) > I$$

3.2 Part (b)

Let us define the expected profits of the firm if it undertakes the investment in period 2 $\Pi\left(2\right)$. Then we can derive that

$$\Pi\left(2\right) = \Pr\left(\pi_2 > I\right) E\left(\pi_2 - I \mid \pi_2 > I\right)$$

Solutions PS4 Macro III

1 Two-Sided Lack of Commitment: Stationary Allocations

We have

$$\begin{aligned} V^1 &= u\left(c^1\right) + \beta\left[pV^1 + (1-p)\,V^2\right] \\ V^2 &= u\left(c^2\right) + \beta\left[pV^2 + (1-p)\,V^1\right] \end{aligned}$$

clearly, $V^2(y,x) = V^1(x,y)$.

Add the two conditions to get

$$V^1 + V^2 = u\left(c^1\right) + u\left(c^2\right) + \beta\left[V^1 + V^2\right] \Leftrightarrow$$

$$V^1 + V^2 = \frac{1}{1-\beta}\left[u\left(c^1\right) + u\left(c^2\right)\right].$$

Plug this in the equation for V^1 to get

$$V_1 = u\left(c^1\right) + \beta p V_1 + \beta\frac{1-p}{1-\beta}\left[u\left(c^1\right) + u\left(c^2\right)\right],$$

and rearranging we get to the desired expression

$$\begin{aligned} V^1\left(c^1, c^2\right) &= \frac{1}{1-\beta}\left\{\omega u\left(c^1\right) + (1-\omega)\,u\left(c^2\right)\right\} \\ &\quad \text{where } \omega = \frac{1-\beta p}{1+\beta-2p\beta} > \frac{1}{2} \end{aligned}$$

Notice that $\frac{\partial\omega}{\partial p} = \frac{-\beta(1+\beta-2p\beta)+2\beta(1-\beta p)}{(1+\beta-2p\beta)^2} = \frac{-\beta^2+\beta}{(1+\beta-2p\beta)^2} > 0$ and $\frac{\partial\omega}{\partial\beta} = \frac{-p(1+\beta-2p\beta)+(1-2p)(1-\beta p)}{(1+\beta-2p\beta)^2} = \frac{p-1}{(1+\beta-2p\beta)^2} < 0$. This is intuitive. For β close 1, we don't discount the future at all so V^1 is (almost) an average with weights .5 of both utility levels. When $\beta = 0$ we don't care about the future and thus $V^1 = u(c^{1)}$. the higher β, the more we care about the future and $u(c^2)$ plays a higher role relative to $u(c^1)$ in determining V^1.An the higher the most likely we stay on the same state and thus $u(c_1)$ has a higher weight.

(b) We call a stationary symmetric allocation **feasible** if it satisfies the resource and participation constraints:

$$c^1 + c^2 = e$$

$$V^1\left(c^1, c^2\right) \geq V^1\left(y^1, y^2\right) \quad (1)$$
$$V^2\left(c^1, c^2\right) \geq V^2\left(y^1, y^2\right) \quad (2)$$

Notice that autarky is always feasible.

If (1) holds

$$\frac{1}{1-\beta}\left\{\omega u\left(c^1\right) + (1-\omega)\, u\left(c^2\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right)\right\}$$
$$\Leftrightarrow u\left(y^1\right) - u\left(c^1\right) \leq -\frac{1-\omega}{\omega} u\left(y^2\right) - u\left(c^2\right).$$

For (2) to hold we need (using the fact that $V^2\left(c^1, c^2\right) = V^1\left(c^2, c^1\right)$ and $V^2\left(y^1, y^2\right) = V^1\left(y^2, y^1\right)$)

$$\frac{1}{1-\beta}\left\{\omega u\left(c^2\right) + (1-\omega)\, u\left(c^1\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^2\right) + (1-\omega)\, u\left(y^1\right)\right\}$$
$$\Leftrightarrow u\left(y^1\right) - u\left(c^1\right) \leq -\frac{\omega}{1-\omega} u\left(y^2\right) - u\left(c^2\right).$$

Notice that $\frac{\omega}{1-\omega} > \frac{1-\omega}{\omega}$ as $\omega > .5$, which implies that whenever (1) holds, (2) holds with strict inequality (this is true as long as $c^1 \leq y^1$, see discussion below. If $c^1 > y^1$, then the opposite argument holds).

(c) Full risk sharing $\Leftrightarrow c^1 = c^2 = \frac{e}{2}$ which is clearly feasible. From (b) we know that we only need to check (1). (1) is satisfied if

$$\left\{\omega u\left(\frac{e}{2}\right) + (1-\omega)\, u\left(\frac{e}{2}\right)\right\} \geq \frac{1}{1-\beta}\left\{\omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right)\right\}$$
$$\Leftrightarrow u\,(e/2) \geq \omega u\left(y^1\right) + (1-\omega)\, u\left(y^2\right) \quad (3)$$

(d) A higher p and a lower β both increase ω. The higher is ω, the higher is the autarky value when you have the high pay-off, which makes harder for the full-risk sharing equilibrium to be feasible.

For the next part, we can write the condition as

$$\Leftrightarrow u\,(e/2) \geq \omega u\,(e/2 + b) + (1-\omega)\, u\,(e/2 - b)\,,$$

and take second order Taylor approximations around $e/2$ on the RHS to get

$$\begin{aligned} u\,(e/2) \geq\ & \omega u(e/2) + \omega u'(e/2)b + \frac{w}{2}b^2 u''(e/2) + \\ & (1-\omega)\, u\,(e/2) - (1-\omega)u'(e/2)b + \frac{1-w}{2}b^2 u''(e/2) + O(3) \end{aligned}$$

$$\Leftrightarrow u(e/2) \geq u(e/2) + (2\omega - 1)u'(e/2)b + \frac{1}{2}b^2 u''(e/2) \Leftrightarrow$$

$$(2\omega - 1)u'(e/2) \leq -\frac{1}{2}bu''(e/2),$$

and for b small enough (i.e. taking the limit when $b \to 0$)

$$(2\omega - 1)u'(e/2) \leq 0.$$

which is a contradiction. So for b small enough perfect risk sharing is not possible.

To proof the same thing for σ small enough, take the same expression

$$(2\omega - 1)u'(e/2) \leq -\frac{1}{2}bu''(e/2),$$

$\Leftrightarrow$

$$(2\omega - 1) \leq \frac{-\frac{1}{2}bu''(e/2)}{u'(e/2)}$$

and use the utility function to get

$$(2\omega - 1) \leq \frac{b\sigma}{e}$$

and taking the limit when $\sigma \to 0$

$$(2\omega - 1) \leq 0,$$

which again is a contradiction.

(e) We want to proof that the best symmetric allocation satisfies

$$c^1 + c^2 = e$$

$$\omega\left[u\left(c^1\right) - u\left(y^1\right)\right] + (1-\omega)\left[u\left(c^2\right) - u\left(y^2\right)\right] = 0$$

and $y^2 \leq c^2 \leq c^1 \leq y^1$ (i.e. satisfies the above two equations and has less variability than autarky).

Assume $c^1 \leq y^1$. Notice that if full risk sharing is not binding then $IC1$ must be binding, i.e. $\omega\left[u\left(c^1\right) - u\left(y^1\right)\right] + (1-\omega)\left[u\left(c^2\right) - u\left(y^2\right)\right] = 0$. Can we have $c^2 > c^1$? No,. if full risk sharing is not attainable, neither it is an allocation with $y^2 \leq c^1 \leq c^2 \leq y^1$. To see that notice that

$$\omega u(e/2 - c) + (1-\omega)u(e/2 + c) < u(e/2) < \omega u(e/2 + b) + (1-\omega)u(e/2 - b),$$

where the second inequality comes from full risk sharing not being attainable and the first from the fact that $\omega > .5$ and concavity of u.

Can we have $c^1 > y^1$? No because an allocation like this will give the same expected payoff than autarky with more volatility, and thus autarky is a better solution.

(f) Here we compute numerically the optimal allocation for the case where the utility function is of the CRRA form: $u(c) = c^{1-\sigma}/(1-\sigma)$.

Use the following parameters[1] $\beta = .65$, $p = 0.75$, $y^1 = 0.641$ and $y^2 = 0.359$. Plot the optimal c^1 and c^2 as functions of σ for the range $\sigma \in [1, 5]$ (i.e. use a grid over σ with enough points between 1 and 5) [2].

See graphs.

2 Risk Free rate Puzzle

(a)See graphs attached.

There are two forces counteracting here. For high levels of γ, the elasticity of substitution is really low and we need really high levels of the risk free rate to convince the individual to save enough to achieve the required growth rate in consumption. on the other hand, high γ means high risk aversion and thus as $u''' > 0$, a stronger precautionary savings motive. This last effect makes individuals save more for any level of the interest rate, and therefore we require a lower one to achieve the desired growth rate in consumption. To see that clearly, see how when the precautionary savings motive is absent ($\text{Var}_c = 0$) the interest rate is increasing in γ. And for enough variance, the second effect dominates the first and we get a decreasing pattern.

(b) This graph reflects how the second effect dominates in this case for γ big enough. Notice that we need $\gamma > 30$ to be able to explain the value of the risk free rate.

(c) This again reflect the same facts. And it makes things worst as we are lowering the variance of consumption.

[1] These parameters imply a standard deviation for log-output of .29 and a first-order autocorrelation of .5, matching findings by Heaton and Lucas (1996) using the PSID.

[2] Hint: Make sure you first check for perfect risk sharing. If full risk sharing is available take that allocation. Otherwise compute the allocation that satisfies the requirements in part (e), which may imply autarky or some insurance (watch out: do not compute an allocation with more variability than autarky!).