

Solutions for Problem Set #3

Macro 14.453

October 5, 2006

1 Precautionary Savings in General Equilibrium

Let utility be given by:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

where $u(c) = -\frac{1}{\gamma} \exp\{-\gamma c\}$. Assume the standard intertemporal budget constraint

$$A_{t+1} = (1+r)(A_t + y_t - c_t). \quad (2)$$

(note: we do not necessarily impose $\beta(1+r) = 1$ here). Assume that y_t is *i.i.d.* across time and agents. Let $y_t = \bar{y} + \varepsilon_t$ where ε_t is *i.i.d.* and $E_{t-1}\varepsilon_t = 0$. We do not impose a borrowing constraint on this problem, A_t can take any value, although a no-Ponzi condition should be thought as being implicitly imposed for the problem to be well defined (you will not have to think about this no-Ponzi condition explicitly for solving the problem though).

1.1 Part (a)

Show that the consumption function,

$$c_t = \frac{r}{1+r} \left[A_t + y_t + \frac{1}{r} \bar{y} \right] - \pi(r, \gamma) \quad (3)$$

for some $\pi(r, \gamma)$ implies,

$$\Delta c_t = \frac{r}{1+r} [y_t - \bar{y}] + r\pi(r, \gamma) \quad (4)$$

[note that the functional form of the consumption function, as a function of A_t and current and expected income, is like the CEQ-PIH except for the constant $\pi(r, \gamma)$]

We can lag one period the intertemporal budget constraint (2), *i.e.*

$$A_t = (1+r)(A_{t-1} + y_{t-1} - c_{t-1})$$

and plug it into the consumption function (3). So we obtain

$$c_t = r(A_{t-1} + y_{t-1} - c_{t-1}) + \frac{r}{1+r} \left[y_t + \frac{1}{r} \bar{y} \right] - \pi(r, \gamma)$$

which implies

$$\Delta c_t = r(A_{t-1} + y_{t-1}) - (1+r)c_{t-1} + \frac{r}{1+r} \left[y_t + \frac{1}{r} \bar{y} \right] - \pi(r, \gamma) \quad (5)$$

Then we can lag one period the consumption function (3) and multiply it by $(1+r)$ obtaining

$$(1+r)c_{t-1} = r \left[A_{t-1} + y_{t-1} + \frac{1}{r} \bar{y} \right] - (1+r)\pi(r, \gamma) \quad (6)$$

and pluggin together (5) and (6) we get exactly

$$\Delta c_t = \frac{r}{1+r} [y_t - \bar{y}] + r\pi(r, \gamma)$$

which is what we wanted to prove.

1.2 Part (b)

Use the Euler equation and your results in (a) to show that the consumption function in (a) is optimal for some π (hint: use the Euler equation to guess and verify the optimality of the above consumption function) which depends on r and the distribution of ε .

From solving our optimization problem, as we saw in class, we can get the standard Euler equation

$$u'(c_t) = \beta(1+r) E_t[u'(c_{t+1})]$$

Using our preferences specification, $u(c) = -\frac{1}{\gamma} \exp\{-\gamma c\}$, we have that

$$u'(c) = \exp\{-\gamma c\}$$

and so we can rewrite our Euler equation as

$$\exp\{-\gamma c_t\} = \beta(1+r) E_t[\exp\{-\gamma c_{t+1}\}]$$

which implies

$$1 = \beta(1+r) E_t[\exp\{-\gamma \Delta c_t\}]$$

Thus, we can use the expression (4) we proved in point (a) and rewrite the Euler equation as

$$1 = \beta(1+r) E_t \left[\exp \left\{ -\gamma \left(\frac{r}{1+r} \varepsilon_t + r\pi(r, \gamma) \right) \right\} \right]$$

and so

$$\exp\{\gamma r \pi(r, \gamma)\} = \beta(1+r) E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_{t+1} \right\} \right]$$

and taking log

$$\pi(r, \gamma) = \frac{1}{\gamma r} \left[\log \beta(1+r) + \log E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_{t+1} \right\} \right] \right] \quad (7)$$

By construction this means that we can guess the following consumption function

$$c_t = \frac{r}{1+r} \left[A_t + y_t + \frac{1}{r} \bar{y} \right] - \frac{1}{\gamma r} \left[\log \beta(1+r) + \log E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_{t+1} \right\} \right] \right]$$

or, equivalently

$$\Delta c_t = \frac{r}{1+r} [y_t - \bar{y}] + \frac{1}{\gamma} \left[\log \beta(1+r) + \log E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_{t+1} \right\} \right] \right]$$

and plug it back into our Euler Equation to verify it.

1.3 Part (c)

Show that $\pi(r, \gamma) > 0$ if r is such that $\beta(1+r) = 1$. Compare this to the CEQ-PIH case. How does π depend on the uncertainty in y_t ?

Note that if $\beta(1+r) = 1$ from equation (7) we have

$$\pi(r, \gamma) = \frac{1}{\gamma r} \log E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_{t+1} \right\} \right]$$

By Jensen's inequality we know that

$$\log E(x) > E(\log x)$$

so that we can derive

$$\pi(r, \gamma) > \frac{1}{\gamma r} E_t \left(-\frac{\gamma r}{1+r} \varepsilon_t \right) = \frac{1}{1+r} E_t(\varepsilon_t) = 0$$

Note that

$$\Delta c_t = \frac{r}{1+r} [y_t - \bar{y}] + r\pi(r, \gamma) \quad (8)$$

differs from the CEQ-PIH case only exactly because of $\pi(r, \gamma)$. If markets are complete but there is uncertainty, consumption turns out to be more increasing because of the precautionary savings motiv. Note that as $\exp(\cdot)$ is a convex function, a mean preserving spread of ε_t is going to increase $E_t \left[\exp \left\{ -\frac{\gamma r}{1+r} \varepsilon_t \right\} \right]$ so that we can conclude that $\pi(r, \gamma)$ is increasing in the uncertainty of ε_t , or equivalently of y_t .

Think at this simple example to be convinced that the expected value of a convex function is increasing in uncertainty. Imagine to have two variables:

$$\begin{aligned} \varepsilon &= \varepsilon_0 && \text{with probability } 1 \\ \tilde{\varepsilon} &= \begin{cases} \varepsilon_1 & \text{with probability } p_1 = \frac{1}{2} \\ \varepsilon_2 & \text{with probability } p_2 = \frac{1}{2} \end{cases} \end{aligned}$$

where

$$E(\varepsilon) = \varepsilon_0 = \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 = E(\tilde{\varepsilon})$$

Suppose $f(\cdot)$ is a convex function then it is clear that

$$E[f(\varepsilon)] = f(\varepsilon_0) < \frac{1}{2}f(\varepsilon_1) + \frac{1}{2}f(\varepsilon_2) = E[f(\tilde{\varepsilon})]$$

Since $\exp(\cdot)$ is a convex function and $\log(\cdot)$ is a monotonic increasing function it turns out from (7) that $\pi(r, \gamma)$ is an increasing function in the uncertainty of y_t as we have stated previously.

1.4 Part (d)

Assume there is a constant measure 1 of individuals in the population. Argue that for aggregate consumption and assets to be constant and finite in the long run (in the limit as $t \rightarrow \infty$) we require that $\pi(r, \gamma) = 0$. What does this imply about average long-run asset holdings as a function of r and γ (denote this by $A(r, \gamma)$)? What is happening to the cross-section of consumption? Does this distribution converge?

To require aggregate consumption and assets to be constant and finite in the long run is equivalent to require that variations in aggregate consumption and assets to be zero as $t \rightarrow \infty$. Let us define $F(i)$ the probability distribution across households so that from the result in part (a)

$$\Delta c_t^i = \frac{r}{1+r} \varepsilon_t^i + r\pi(r, \gamma)$$

and taking the average

$$\Delta C_t = \int_0^1 \Delta c_t^i dF(i) = \frac{r}{1+r} \int_0^1 \varepsilon_t^i dF(i) + r\pi(r, \gamma)$$

Since there is no aggregate uncertainty, using the law of large number we can infer that $\int_0^1 \varepsilon_t^i dF(i) = 0$ and then

$$\Delta C_t = r\pi \tag{9}$$

Taking the equation for assets accumulation and replacing the consumption function we get

$$A_{t+1}^i - A_t^i = \varepsilon_t^i + (1+r)\pi(r, \gamma),$$

and taking the average

$$\begin{aligned} \Delta A_t &= A_{t+1} - A_t = \int_0^1 A_{t+1}^i dG(i) - \int_0^1 A_t^i dG(i) = \\ &= \int_0^1 \varepsilon_t^i dF(i) + (1+r)\pi(r, \gamma) \end{aligned}$$

so that, using again the law of large numbers, we get

$$\Delta A_t = (1+r)\pi(r, \gamma) \tag{10}$$

Equations (9) and (10) imply that in order to have finite and constant aggregate consumption and assets $\forall t$, and so also for $t \rightarrow \infty$, we need $\pi(r, \gamma) = 0$.

This means that $A(r, \gamma)$ is a discontinuous function in (r, γ) . In fact average long run asset holdings are going not to diverge only for those combinations of r and γ such that $\pi(r, \gamma) = 0$. In fact for all the combinations of (r, γ) such that $\pi(r, \gamma) > 0$, we have that $A(r, \gamma) \rightarrow \infty$ and $C(r, \gamma) \rightarrow \infty$ when $t \rightarrow \infty$ as we can see easily from equations (9) and (10). Viceversa, for all the combinations of (r, γ) such that $\pi(r, \gamma) < 0$, we have that $A(r, \gamma) \rightarrow -\infty$ and $C(r, \gamma) \rightarrow -\infty$ when $t \rightarrow \infty$ (since here no borrowing constraints are imposed and consumption can be negative).

Finally, note that the cross-section distribution of consumption does not converge in our model, at odds with the standard Aiyagari model. This crucial difference is coming from our specification of preferences that is characterized by a constant absolute risk aversion, but an increasing relative risk aversion, joint with the fact that we are not assuming any borrowing constraint here, neither exogenous or endogenous (such as the natural borrowing constraint that is driven from assuming non negative consumption). Recall that in class Ivan showed the two crucial characteristics of the graph for the evolution of the cash-in-hands in the Aiyagari model that are required to get convergence in the cross-section distributions of x , and so of assets and consumption. The first one was that the cash-in-hands (deriving from the policy function for assets) after the highest realization crosses at a finite point the 45° line and the second one was that the cash-in-hands (deriving from the policy function for assets) after the lower realization crosses the 45° line in its flat part. Note that we could prove that these two facts apply in the Aiyagari specification, but the two proves cannot be used in our specification. The first one required not exploding relative risk aversion and the second one that there is a borrowing constraint in order to have a flat part of the function. Both these characteristics are not matched from our model's specification and so we cannot prove that the distribution converges. In fact we can easily see that it does not: although the aggregate is well defined, the process for each individual's consumption follows a random walk

$$c_{t+1} = c_t + \frac{r}{1+r} \varepsilon_{t+1} \quad (11)$$

We know that if there was convergence, the cross section ergodic distribution of consumption would have been equal (from the law of large numbers) to the time-series ergodic distribution of consumption for a single individual. But since individual consumption follows a random walk, clearly the distribution of consumption for a single individual over time cannot converge, and so

cannot the cross-section one!

1.5 Part (e)

From the condition $\pi = 0$ we have

$$0 = \frac{1}{\gamma r} \left[\log \beta (1 + r) + \log E_t \left[\exp \left\{ -\frac{\gamma r}{1 + r} \varepsilon_{t+1} \right\} \right] \right] \quad (12)$$

Moreover from the normality assumption we have

$$E_t \left[\exp \left(-\frac{\gamma r}{1 + r} \varepsilon_{t+1} \right) \right] = \exp \left[\frac{1}{2} \left(\frac{\gamma r}{1 + r} \right)^2 \sigma_\varepsilon^2 \right]$$

The complete markets equilibrium interest rate is

$$r = \frac{1 - \beta}{\beta} \approx 3.093\%$$

Under incomplete markets we can calculate the equilibrium interest rate when $\gamma = 1$ from the following equation:

$$0 = \log \beta (1 + r) + \frac{1}{2} \left(\frac{r}{1 + r} \right)^2 \sigma_\varepsilon^2$$

We can solve it numerically (using any program you want: Maple, excel,...) and get

if $\sigma = 0.2$ we get

$$r^e \approx 3.091\%$$

if $\sigma = 0.2$ we get

$$r^e \approx 3.085\%$$

Note that as expected the equilibrium interest rate is lower the higher is the risk aversion. In fact if individuals are willing to save more for precautionary motiv, to keep the markets clear, for a given demand we need a lower interest rate to counterbalance this willingness. Of course the same reasoning applies when we compare the interest rate for any risk aversion degree

to the one in a world of certainty where there is no room for precautionary savings and the interest rate turns out to be necessarily higher.

(f) Briefly discuss how you would think of calibrating γ if you really believed that preferences are CRRA $c^{1-\sigma}/(1-\sigma)$ for some known value of σ , but you want to work with this model as an approximation (because of its analytical tractability).

Speculate on whether this is likely to be a good approximation for learning about r^e . Can this be a good approximation for learning about the long-run (invariant) distribution of asset holdings?

This question was meant to make you think about all these issues a little bit more. There is no correct answer, but many possible ones according to what you think is more important an economic model like this focus on.

A possibility to calibrate γ could be to use the fact that we know that with our preferences specification, the relative risk aversion coefficient is equal to

$$RRA = -\frac{u''(c)c}{u'(c)} = \gamma c$$

We could then use estimates for the constant relative risk aversion coefficient, σ , derived from using the CRRA preferences and dividing it by a constant value for consumption that could be the consumption level in steady state for example (people suggested also to use the mean).

What is important to notice is that there is a trade-off coming from using CARA preferences as an approximation. In fact it could be useful for learning about r^e since in equilibrium aggregate consumption is constant as we derived previously, but it is definitely not a good approximation for learning about the long-run distribution of asset holdings, since, as we discuss, there is not a long-run invariant distribution of assets!

2 Income Fluctuation Problem – Numerical Computation

This exercise asks you to compute numerically an income fluctuations problem. The problem is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to:

$$x_{t+1} = (1 + r)(x_t - c_t) + y_{t+1}$$

where x_t represents “cash in hand” and y_t labor income. The income process is assumed to be i.i.d. with only two possible realizations $y_l = 1.5w$ and $y_h = .5w$ with $1/2$ probability each. Thus, aggregate labor supply is normalized to 1. Set $\beta = .96$, $r = 2\%$ and $w = 1$.

You are given the basic matlab codes that should allow you to compute the solution to the income fluctuation problem

Perform all calculations below for $\sigma = .75$ and $\sigma = 2$. Since you have been given the basic code do not hand in the Matlab code. Instead, stress the intuition for the results you obtain.

We solve this problem by iterating on the Bellman equation

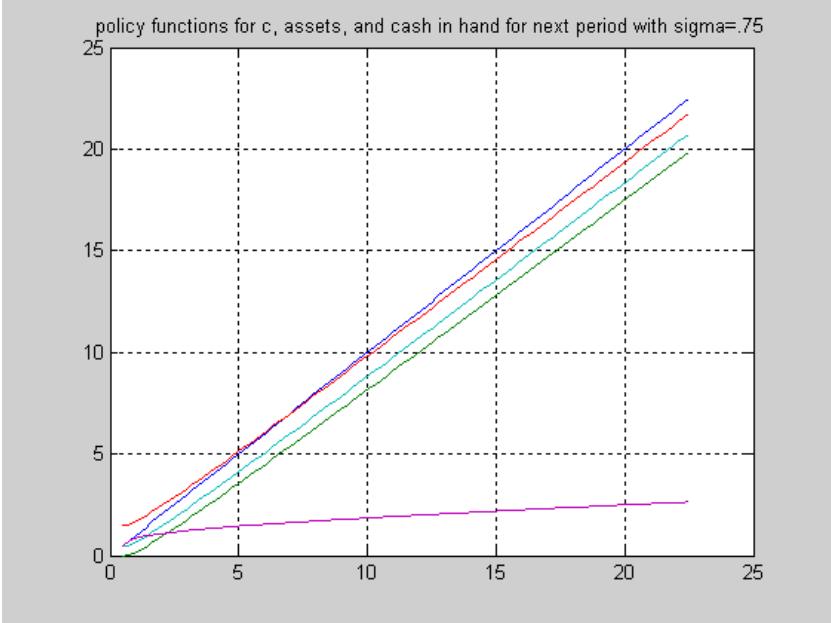
$$v(x) = \max_{0 \leq c \leq x} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta E v((1+r)(x-c) + y) \right\}$$

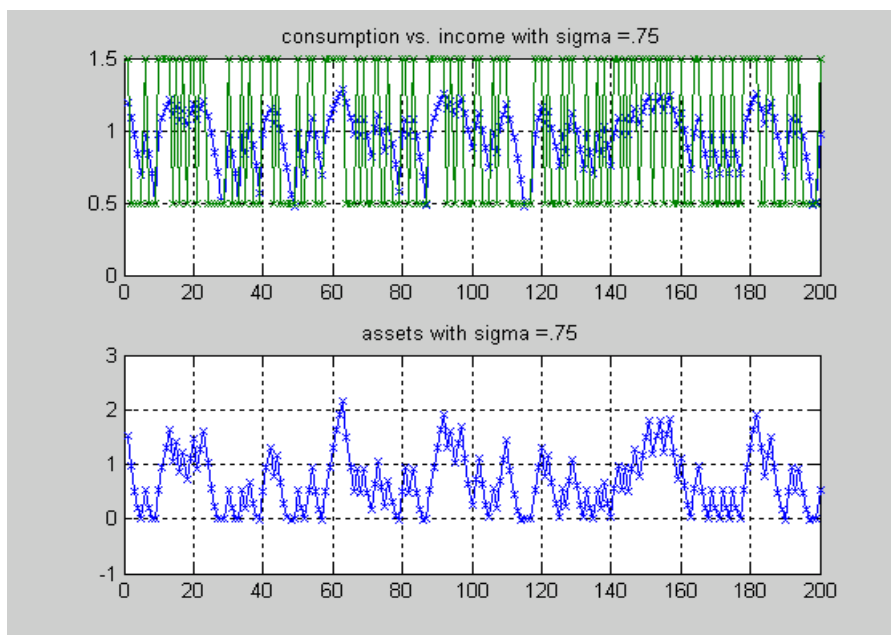
starting from a decent guess (like the one obtained from consuming all future current income and the interest on current cash in hand – so that x_t remains constant).

2.1 Part (a)

Solve the optimal consumption problem obtaining the consumption function $c(x)$. Plot the function for consumption, asset holdings $a'(z) = z - c(z)$, and cash-in-hand for tomorrow (for both realizations of tomorrow’s income shock), that is, $z'(a, y') = (1+r)a'(z) + y'$ for both possible values of y' .

Look at pictures (1) and (2). Note that the graph behaves exactly as expected so that the cash-in-hand function derived from the policy function for assets and the high realization of the shock crosses the 45° line at a finite time and the cash-in-hand function after the low realization of the shock crosses it in its flat part. These characteristics are crucial to have an ergodic distribution of assets cross-section which is derived as the limiting invariant distribution of assets of a single individual over time. This is used in the following part: we evaluate the cross-section average assets as the mean of the time-series assets of a single individual for a random realization of a sequence of shocks long enough (in order to use the law of large numbers).



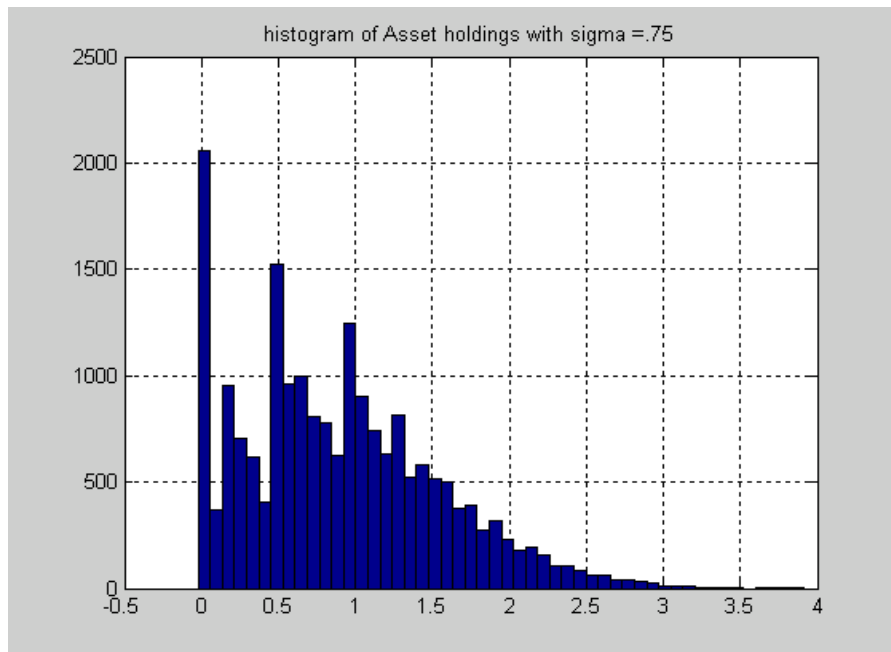
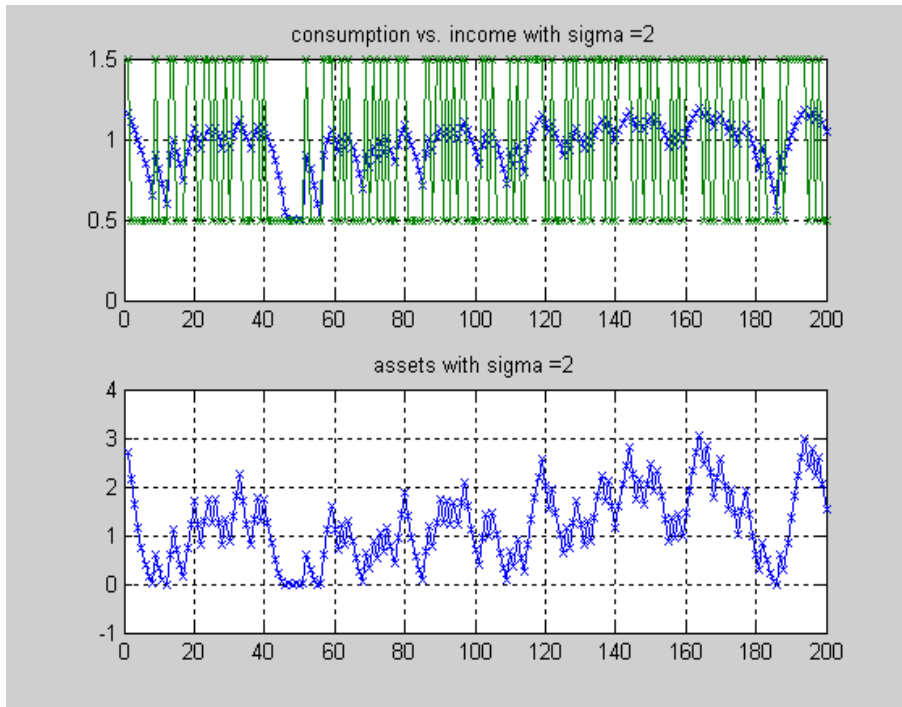


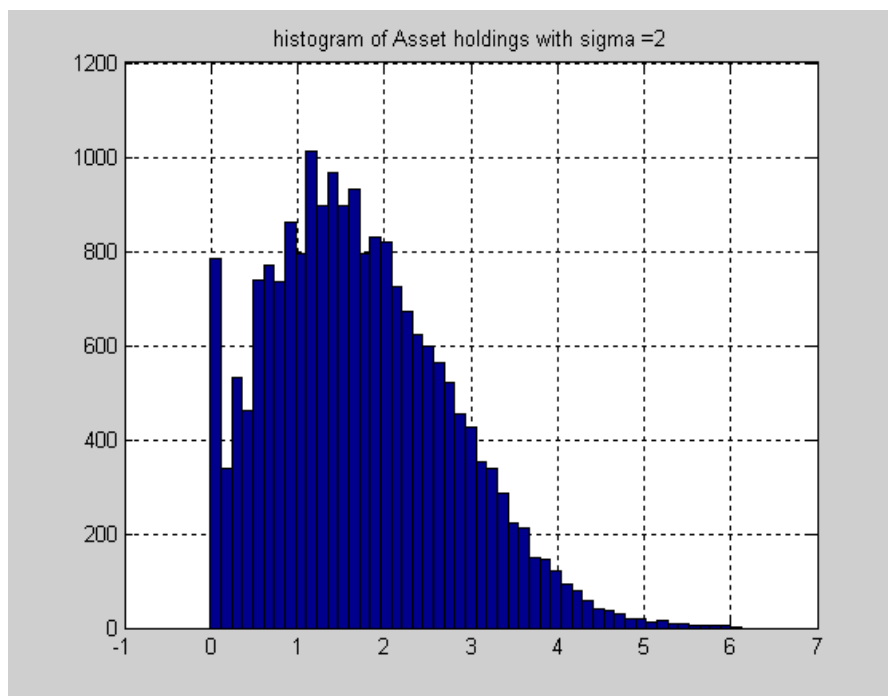
2.2 Part (b)

Use this policy function together with random shocks to simulate the evolution of cash-in-hand, income, consumption and assets for 200 periods. Plot the simulated series for consumption, income and assets. Is consumption smoother than income? How high are asset holdings? For what fraction of periods is the agent liquidity constrained (i.e. $x_t - c_t = 0$)? How do your results depend on σ ?

We can see from the graphs (3) and (4) that consumption is smoother than income, because of the precautionary savings motif. In fact for higher risk aversion we can note that consumption is much smoother and the level of asset holdings is higher on average. Moreover, irrespective of the random shock realization specific to your simulation, the fraction of periods in which the agent is liquidity constrained is generally lower for an higher level of risk aversion (it can perfectly happen that your specific simulation gives you that the agent is never constrained neither for the case of $\sigma = .75$).

Note that the histograms (5) and (6) helps to check that the program we are running is correct. In fact we are bounding the cash-in-hands from above to get a more efficient program, but we are allowed to do that only





because we can see from the histograms of cash-in-hands that there is not positive mass of individuals with cash-in-hands above a certain value much lower than the bound that we are imposing. That means that the bound that we add is not binding and, thus, does not change the results.

2.3 Part (c)

Compute a longer simulation (around 11001 periods), throw away the first 1001 periods (that is, from $t = 0$ to $t = 1000$ included) and calculate the average asset holdings with the remaining periods. That is, based on the simulated sequence for $\{z_t\}_{t=0}^{11000}$ compute $\frac{1}{10000} \sum_{t=1001}^{11000} a'(z_t)$ where $a'(z)$ is the policy function found in a .

The average assets turn out to be

if $\sigma = .75$

$$A \approx 1.1609$$

if $\sigma = 2$

$$A \approx 2.0257$$

2.4 Part (d)

(Carroll, 1997) Modify the income process to have the following characteristic: there is a small probability π of income being zero. If this event does not occur income is drawn from the same distribution as before.

Argue that, with the preferences above, the borrowing constraint will never bind: we always have $a_t = x_t - c_t > 0$ (you do not have to compute. If we allow for some borrowing, so that we replace the constraint $a_t \geq 0$ with $a_t \equiv x_t - c_t \geq -b$ for some positive $b > 0$, argue that this condition will never bind and that in fact $a_t > 0$.

This comes from the fact with this shock specification, there is always a strictly positive probability, even if infinitesimal, that the individual turns out to have 0 consumption, if the borrowing constraint binds. In fact in this case, the agent has no assets, and there is a strictly positive probability that he receives $y = 0$ so that he can consume only 0. But we know that

$$\lim_{c \rightarrow 0} u'(c) = \infty$$

so it cannot be that the Euler condition holds if we have this happening. In fact the agent would always be willing to sacrifice 1 unit of consumption today to save and get an infinite utility tomorrow. That is why the borrowing constraint will never bind.

2.5 Part (e)

Now you will compute the equilibrium for this model as in Aiyagari (1994). Using the Cobb-Douglas technology $F(K, L) = K^\alpha L^{1-\alpha}$ solve the following system for $K(r)$ and $w(r)$:

$$\begin{aligned} r &= F_k(K, 1) - \delta \\ w &= F_L(K, 1) \end{aligned} \tag{13}$$

Here $K(r)$ and $w(r)$ represent the level of capital demanded by firms and the associated wage if the interest rate at a steady state were equal to r and labor supply and demand are equal.

For any proposed value of r (equivalently one can propose a value for K and obtain the implied proposed r using (13)) we can use the implied value for the wage $w(r)$ and solve the individual's income fluctuations problem given r and $w(r)$. Then using the method in (c) we can calculate the average asset holdings, denote this by $A(r)$. Think of $A(r)$ as representing the steady state supply of capital by individuals when the interest rate equals to r and the wage is $w(r)$.

We are looking for a value for r such that $K(r) = A(r)$, i.e. that the quantity demanded and supplied of capital at a steady state are equal. Using $\alpha = .33$, $\delta = 8\%$ and the parameters given above plot $A(r)$ against $K(r)$ by computing $A(r)$ for several values of r (pick values in the interval $0 \leq r < 1/\beta - 1$). Find the value of r and K at which both curves intersect.

From (13) and our Cobb-Douglas specification we can calculate the capital and the wage as a function of r as follows

$$K(r) = \left(\frac{\alpha}{r + \delta} \right)^{\frac{1}{1-\alpha}} \quad (14)$$

$$\omega(r) = (1 - \alpha) \left(\frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1-\alpha}} \quad (15)$$

Moreover from the income-fluctuation problem we have solved in part (a) to (c) we can derive the average asset holdings $A(r)$ as a function of r . Note that to calculate $A(r)$ for a given r you have to slightly modify the program to adjust the wage using equation (15) which was normalized to 1.

As specified in the question, In order to find the general equilibrium values of r and K we need to find the value of r such that the markets clear, *i.e.* the demand and the supply of assets are equalized

$$K(r) = A(r)$$

We can proceed with two methods.

1. You can create a grid for r and repeat the program for any value of r to construct $A(r)$. Then compute $K(r)$ for the same grid of r using equation (14) and look for the intersection. This method is correct, but maybe not the most efficient in terms of time.

2. You can use the interpolation technique suggested by Aiyagari, which is maybe more efficient. You can start with $r = 0$ so that you are sure that

$$K(r) > A(r)$$

and then proceed in increasing r with a not fine grid since we know the intersection won't happen for r very close to 0. I proceeded by increasing r by .01. For any value of r evaluate $K(r)$ and $A(r)$ and check if

$$K(r) > A(r)$$

As long as this is true, keep on increasing r by .01. When you reach an r_1 such that

$$K(r_1) < A(r_1)$$

than you start interpolating. Let me be more precise. Let us define r_0 the value of r you checked before r_1 (*i.e.* $r_0 = r_1 - 0.01$) so that you are sure

$$r_0 < r_1$$

Then choose r_2 as

$$r_2 = \frac{r_0 + r_1}{2}$$

and calculate $K(r_2)$ and $A(r_2)$. The following step is to choose r_3 as follows:

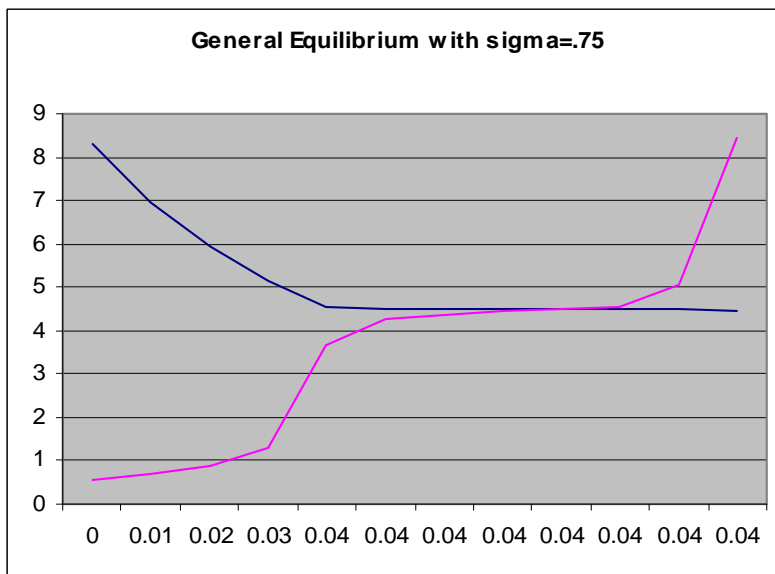
$$\text{if } K(r_2) > A(r_2) \Rightarrow r_3 = \frac{r_2 + r_1}{2}$$

and

$$\text{if } K(r_2) < A(r_2) \Rightarrow r_3 = \frac{r_2 + r_0}{2}$$

Then you proceed forward in this way, averaging the last value of interest rate you used (at this step r_2) with the lower (r_0) or the higher (r_1) of the two used to construct it, according to if your assets (for r_2) turned out to be respectively greater or smaller than capital (for r_2). In fact we know that capital is decreasing in interest rate and assets are increasing in it, so that this method would help us to converge quickly to the value of interest rate for which capital and assets are equalized.

I used the second method (see graphs (7) and (8)) and I got that



if $\sigma = .75$ then

$$r^e = 0.0404$$

if $\sigma = 2$ then

$$r^e = 0.0375$$

Note that , as expected, the equilibrium interest rate turns out to be lower in the case of higher risk aversion because people are more willing to save for precautionary motif and so in order to have clearing markets for a given technology we need to offset this willingness with a lower interest rate.

