14.451 Problem Set 3

Fall 2009

Due on October 15

1 Unique optimal plans and non-unique steady states

This problem addresses a question raised in class: can we have a strictly concave problem (hence, with a unique optimal plan for any initial condition) with multiple steady states?

Consider the objective function

$$F(x, y) = -(x - y)^2.$$

The set of states is $X = R_+$ and the constraint correspondence is simply $\Gamma(x) = X$ for all x.

1. Show that all the assumptions we made in class to show stability of a steady state are satisfied (see lecture notes 7) except one.

2. Show that the steady state is not unique. What is the set of steady states for this problem?

3. Show that starting with any initial condition x_0 there is a unique optimal plan $\{x_t\}_{t=0}^{\infty}$ and the optimal plan converges to *a* steady state.

2 Optimal growth: a closed form example

Consider the optimal growth problem seen in class with the utility function

$$U(c) = \log(c)$$

and the technology

$$f(k) = Ak^{\alpha}.$$

1. Setup the functional equation for this problem. Make the guess that a function of this form
$$V(k) = a + b\log(k)$$
solves the functional equation for some constant scalars a and b. Find a and b.

2. Argue that $V(k)$ in the previous step is the value function for the sequence problem.

3. Characterize the dynamics of capital and consumption. Show that there is a unique steady state with positive capital k^* and this steady state is globally stable. Show that the policy function is differentiable and that $g'(k^*) = \alpha$.

4. Derive $g'(k^*)$ using the methods in section 6.4 (using the Euler equation to characterize the optimal policy near the steady state). Show that you reach the same conclusion as in (3) (if you cannot get an analytical result, check it numerically using matlab to find eigenvalues and eigenvectors of the matrix M for different values of α).

3 Pricing iPhones

A seller has a fixed stock of a durable good, say iPhones, to sell. The stock of iPhones is already produced, so the only remaining objective is to maximize the present value of revenues.

The seller faces a continuum of consumers with heterogeneous reservation values v uniformly distributed on $[0, 1]$. Each consumer is myopic and buys one unit of the product whenever the price is below his valuation. As the good is durable, consumers buy at most once, so after buying they exit. The firm must decide a pricing strategy, that is, a sequence of prices $\{p_t\}$.

1. Set up the dynamic optimization problem recursively. That is, define the state variable for this problem and write down the Bellman equation.

2. Now define the problem in sequential form. Which came easier for you, the recursive or sequential version?

3. Guess and verify the solution. How does it vary with β? What happens as $\beta \to 1$?

4. Reinterpret the dependence of β as a shortening of the length of a period between price revisions. That is, renormalize your results so that you compute the rate of change in prices per unit of time under this interpretation. Are the results in (3) reasonable?

4 Cyclical paths?

Consider a quadratic dynamic programming problem with a l-dimensional state $x_t \in X = R_+^l$. The constraint correspondence is simply $\Gamma(x) = X$ for all x. The objective function is a general quadratic function and is strictly concave.

Suppose the problem has a unique steady state at $x^* > 0$ and all the assumptions made in class are satisfied so that this steady state is stable.

1. Suppose $l = 1$, is it possible to have the optimal plan $\{x_t^*\}$ converge to x^* along a *cyclical* path, i.e., can the matrix M have 1 complex eigenvalue smaller than 1 in absolute value?

2. Now consider the case $l = 2$. Suppose

$$F_{xx} = \begin{bmatrix} -16 & 5 \\ 5 & -34 \end{bmatrix}; F_{yy} = \begin{bmatrix} -19 & -3 \\ -3 & -8 \end{bmatrix}; F_{xy} = \begin{bmatrix} 9 & -5 \\ 15 & 14 \end{bmatrix}.$$

Show analytically and graphically that the optimal adjustment path to x^* is cyclical (e.g. start at $x_0 = x^* + (4, 1)'$).

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