EXAM 2

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Instructions: This exam is closed-book and closed-notes. You may use a simple calculator. Please read through the exam in order to ask clarifying questions and to allocate your time appropriately. You **must show** all your calculations. You have 90 minutes to complete the exam (90 points). Good Luck!

Question 1 (24 points)

A. (12 points, 3 points each) True, False, or Uncertain. Please avoid long explanations; but no explanation no points.

(i) The result $E[g(X)] = g[E(X)]$ holds for $g(X) = 2X + 4$ and for $g(X) = X^2$.

(ii) Let X be a random variable and define H as $H = h(X)$. Then,

$$\int_{-\infty}^{\infty} h(x)f_X(x)dx > \int_{-\infty}^{\infty} hf_H(h)dh$$

(iii) If two variables are uncorrelated $[\rho(X,Y) = 0]$, then they are independent. For example, the Bivariate Normal Distribution: when both random variables are uncorrelated they are also independent.

(iv) Let $Y = h(X)$, where X is a continuous random variable. It is possible to use the 1-step method to compute the pdf of Y only if $h(X)$ is strictly monotone (increasing or decreasing).

B. (12 points) Let Z_1 and Z_2 be discrete random variables with pmf:

$f(z_1, z_2)$	$z_2 = 0$	$z_2 = 1$
$z_1 = 0$	.1	.4
$z_1 = 1$	.3	.2

(i) (6 points) Compute the expectation of Z_1, the expectation of $Z_1|Z_2 = 0$, and the expectation of $Y = Z_1 + Z_2$.

(ii) (6 points) Compute the variance of $Z_1|Z_2 = 0$ and the covariance between Z_1 and Z_2.

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Question 2 (30 points)

A. (16 points) Let X, Y, and Z be random variables with pdfs $f_X(x)$, $f_Y(y)$, $f_Z(z)$, and joint pdfs $f_{X,Y}(x,y)$, $f_{Z,Y}(z,y)$, $f_{X,Z}(x,z)$, and $f_{X,Y,Z}(x,y,z)$. Let a,b,c, and d be constants. Show the following propositions are true.

Note: For part (i) you are expected to use the formal definition of expectation involving integrals and densities; for part (ii) you are expected to use, without proving them, the properties of expected value, variance, and covariance studied in class—except of course for the one you are asked to prove.

(i) (5 points) If X and Y are independent, then

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

(ii) (7 points) If X, Y, and Z are independent, then

$$\text{Var}(aX + Y + c + Z + d) = a^2\text{Var}(X) + \text{Var}(Y) + \text{Var}(Z)$$

(iii) (4 points) Would the above results hold if X and Y are not independent? Explain your answer using your proofs.

B. (14 points) Let X, Y, V, and Z be random variables and a and b be constants.

(i) (6 points) Assume that $E(Y|X) = aX + b$. Express $E(XY)$ as a function only of the first two moments of the RV X.

(ii) (8 points) Assume that $W = V^2 + Z^2$, where V and Z are independent, $Z \sim N(0,1)$, and $V \sim \text{Exponential}(\beta = 1)$. Find the pdf $W|V$.

Question 3 (22 points)

Codelco Inc. produces copper cathodes. The production process results in copper cathodes with a different mineral law. In particular, the resulting mineral law in each copper cathode deviates (in percentage points) from the industry standard according to a normal distribution with mean 0 and standard deviation 15. Assume that the production process of each cathode is independent of each other.

(i) (4 points) A customer of Copper Inc. needs to buy copper cathodes with a mineral law not lower than 10 percentage points below the industry standard. What is the probability that a particular cathode meets this specification?

(ii) (5 points) Following part (i), assume the customer says he is not willing to buy cathodes unless each cathode satisfies his specification with 90% probability. The customer recommends the company to invest in the production process in order to reduce the variability of the cathodes' law. How much should the standard deviation of the production process change (from its current value of 15) to satisfy the customer's conditions?

(iii) (8 points) Following part (ii), assume the company does not change its production process. Assume also that the customer couldn't find another supplier of copper cathodes and that, although he needs only 255, he is thinking of buying enough cathodes from Codelco Inc. in order to be sure that with 99% probability he will have at least 255 cathodes that satisfy his specifications (a mineral law not lower than 10 percentage points below the industry standard). Without solving for the exact number, find the expression that would give you the minimum number of cathodes that the customer needs to buy.

(iv) (5 points) Copper Inc. can also sell its copper cathodes in the London Market, which are priced according to the law of the mineral: a copper cathode of law L has a price of $\frac{3}{2}L^2$ cents of a dollar. What is the expected price of a copper cathode sold at the London Market? Assume for this question that the industry standard is priced at 600 dollars.

Question 4 (14 points)

Let $X_1, ..., X_n$ be independent random variables, where $\mu_i = 1$ and $\sigma_i = 2$. Define Y as $Y = \frac{1}{n} \sum_{i=1}^{n} X_i$.

    (i) (7 points) Using Chebyshev inequality find all the values of n such that the probability that Y is within 1 unit of its mean is at least 99%.

    (ii) (7 points) Assume now that $X_i \sim N(1, 4)$. Find all the values of n such that the probability that Y is within 1 unit of its mean is at least 99%.