

Exam Practice Questions

Problem 1: Consider the game below.

	A	B
A	(2, 1)	(0, 0)
B	(0, 0)	(1, 2)

(a) What is player 1's minimax payoff?

Suppose player 2 chooses action A with probability q and B with probability $1 - q$. Player 1 earns in expectation $2q$ from playing A and $1 - q$ from playing B . To find the minimax payoff, we solve

$$\min_{q \in [0,1]} \max\{2q, 1 - q\} = \frac{2}{3}.$$

Player 2 delivers the worst possible payoff by choosing $q = \frac{1}{3}$, and player 1 is then indifferent between A and B .

(b) Describe the set of all Nash equilibria (pure and mixed).

There are two pure strategy Nash equilibria (A, A) and (B, B) . There is also exactly one mixed strategy Nash equilibria. Using the solution from the last problem, we see that if player 2 chooses action A with probability $\frac{1}{3}$, player 1 is indifferent between the two actions. By symmetry, if player 1 chooses action B with probability $\frac{1}{3}$, player 2 is indifferent between the two actions. The mixed strategy equilibrium involves player 1 choose A with probability $\frac{2}{3}$ and B with probability $\frac{1}{3}$, and player 2 choosing A with probability $\frac{1}{3}$ and B with probability $\frac{2}{3}$.

Suppose player 1 (the row player) is uncertain about player 2's preferences. The actual game is one of the following two games:

	A	B		A	B
A	(2, 1)	(0, 0)	A	(2, -1)	(0, 0)
B	(0, 0)	(1, 2)	B	(0, 0)	(1, 4)

Player 2 learns her preferences before making a choice, while player 1 must make a choice without further information. The common prior is that the first game is played with probability p , and the second with probability $1 - p$.

(c) As a function of p , describe the set of all (pure strategy) Bayes-Nash equilibria.

With two types of player 2, there are exactly 8 pure strategy profiles. Notice for the second type that action B dominates action A . Therefore, we need only consider four possible strategy profiles: $(A, (A, B))$, $(A, (B, B))$, $(B, (A, B))$, and $(B, (B, B))$. The middle two are clearly not best responses for the first type of player 2.

In the profile $(A, (A, B))$, player 2 is clearly playing a best response. Player 1 earns $2p$ from action A and $1 - p$ from action B . It is a best response for player 1 to choose action a if and only if $2p \geq 1 - p$ or $p \geq \frac{1}{3}$. In the profile $(B, (B, B))$, player 1 is clearly playing a best reply, so this is always an equilibrium.

Problem 2: Each player i in a population of size N makes a non-negative contribution $x_i \in \mathbb{R}$ to a public good. If the vector of investments is (x_1, x_2, \dots, x_N) , player i 's payoff is

$$u_i(\mathbf{x}) = 2\sqrt{\sum_{j=1}^N x_j} - x_i.$$

(a) Write down player i 's best response map. What is the total level of investment $\sum_{j=1}^N x_j$ in any Nash equilibrium? What is the efficient level of investment?

Taking the first order condition, we have

$$0 \geq \frac{\partial u_i}{\partial x_i} = \left(\sum_{j=1}^N x_j \right)^{-\frac{1}{2}} - 1 \quad \implies \quad \sum_{j=1}^N x_j \geq 1,$$

with equality whenever player i invests effort. Player i 's best response is therefore

$$x_i = \max \left\{ 0, 1 - \sum_{j \neq i} x_j \right\}.$$

In equilibrium, the total level of investment must be exactly one, since no one invests above this amount in a best reply, and someone will invest 1 if no one else invests.

Define $\bar{x} \equiv \sum_{j=1}^N x_j$. To find the efficient level of investment, we maximize

$$\sum_{i=1}^N u_i(\mathbf{x}) = 2N\sqrt{\bar{x}} - \bar{x}.$$

The first order condition gives

$$0 = \frac{N}{\sqrt{\bar{x}}} - 1 \quad \implies \quad \bar{x} = N^2.$$

The efficient level of investment is much higher than the equilibrium level and increases in with N .

Suppose instead that spillovers from other players' investments are imperfect, and player i 's payoff is

$$u_i(\mathbf{x}) = 2\sqrt{x_i + \delta \sum_{j \neq i} x_j} - x_i$$

for some $\delta \in (0, 1)$.

- (b) Write down player i 's best response map. What is the total level of investment $\sum_{j=1}^N x_j$ in a symmetric Nash equilibrium? What is the efficient level of investment?

The first order condition gives

$$0 \geq \frac{1}{\sqrt{x_i + \delta \sum_{j \neq i} x_j}} - 1 \quad \implies \quad x_i + \delta \sum_{j \neq i} x_j \geq 1,$$

with equality whenever player i invests. Player i 's best response is

$$x_i = \max \left\{ 0, 1 - \delta \sum_{j \neq i} x_j \right\}.$$

In a symmetric strategy profile, we have $x_i = x^*$ for all i . This is an equilibrium if and only if

$$x^* = 1 - \delta(N-1)x^* \quad \implies \quad x^* = \frac{1}{1 + \delta(N-1)},$$

implying a total level of investment

$$\bar{x} = \frac{N}{1 + \delta(N-1)}.$$

The efficient level of investment maximizes

$$\sum_{i=1}^N \left(2\sqrt{x_i + \delta \sum_{j \neq i} x_j} - x_i \right).$$

Taking a derivative with respect to x_i , we get

$$\frac{1}{\sqrt{x_i + \delta \sum_{j \neq i} x_j}} + \sum_{j \neq i} \frac{\delta}{\sqrt{x_j + \sum_{k \neq j} x_k}} - 1.$$

Taking advantage of symmetry, the efficient investment for an individual player x' satisfies

$$\frac{1 + \delta(N - 1)}{\sqrt{(1 + \delta(N - 1))x'}} = 1,$$

implying

$$x' = 1 + \delta(N - 1)$$

and efficient total investment of N times this quantity.

Suppose $N = 2$ and now the game takes place in two stages. In the first stage, players 1 and 2 invest efforts $s_1, s_2 \geq 0$ at constant marginal cost c to establish a relationship. These investments result in the tie strength $\delta(s_1, s_2) = \min\{s_1 + s_2, 1\}$. Once these investments are made, the resulting tie strength is observed, and we move to the second stage. In the second stage, the players invest as above. The payoff to player i is

$$u_i(\mathbf{x}, \mathbf{s}) = 2\sqrt{x_i + \delta(s_i, s_{-i})x_{-i}} - x_i - cs_i.$$

- (c) As a function of c , what tie strength forms in equilibrium? What is the efficient outcome?

We use our result from the last part. Fixing the tie strength, player i invests $1 - \delta(s_i, s_{-i})x_{-i}$ in equilibrium, yielding a total payoff

$$1 + \delta(s_i, s_{-i})x_{-i} - cs_i.$$

The derivative of δ with respect to s_i is exactly x_{-i} if $\delta < 1$ and 0 if $\delta \geq 1$. Therefore, if $c < x_{-i}$, player i will invest up to the point where $\delta = 1$, and if $c > x_{-i}$, player i will choose $s_i = 0$.

We therefore have 2 possible cases. If $\delta = 1$ in equilibrium, then from part (a) we know players are willing to invest up to 1. From our above analysis, in order to have $\delta = 1$ in equilibrium, we must have at least one of the players investing an amount at least c . If $c \leq 1$, we can have such an equilibrium, with possibly asymmetric strategies. If $\delta = 0$ in equilibrium, then both players invest $x_i = 1$. For this to be an equilibrium, we must have $c \geq 1$, otherwise the players would wish to invest in a link. Consequently, we have $\delta = 1$ in any equilibrium with $c < 1$, and $\delta = 0$ in equilibrium if $c > 1$.

In an efficient strategy profile, we get a similar all-or-nothing result on link investment. If $\delta = 1$, it is efficient for each player to invest 2, yielding total welfare $6 - c$. If $\delta = 0$, each player efficiently invests 1, yielding total welfare 2. It is efficient to invest in the link as long as $c \leq 4$.

Problem 3: Consider a variant of the mean-field diffusion model from the first lecture on diffusion. Each agent in a large population chooses between two actions 0 and 1. Agents have degrees drawn independently from the distribution D and private values drawn independently

from a uniform distribution on $[0, 1]$. If an agent has degree d and value v , and a neighbors adopt, the payoff to adoption is

$$u(d, v, a) = av - c.$$

That is, the payoff to adoption increases linearly in the number of neighbors who end up adopting. Recall the neighbor degree distribution \tilde{D} that corrects for the friendship paradox:

$$\mathbb{P}(\tilde{D} = d) = \frac{\mathbb{P}(D = d) \cdot d}{\sum_{k \in \mathbb{N}} \mathbb{P}(D = k) \cdot k}.$$

Time is discrete. Let $\sigma_{t,d}$ denote the fraction of degree d agents adopting at time t . At time $t + 1$, each agent chooses an action to maximize expected utility, assuming that neighbors will each adopt with independent probability

$$\sigma_t = \sum_{d \in \mathbb{N}} \mathbb{P}(\tilde{D} = d) \sigma_{t,d}.$$

- (a) Suppose the degree distribution D takes the value 3 with probability one. Compute $\sigma_{t+1,3}$ as a function of $\sigma_{t,3}$. Find the steady state adoption levels. Which are stable?

First note that having no players adopt is always an equilibrium. If D is 3 with probability one, then so is \tilde{D} . For a player with value v , the expected value of a is just $3\sigma_t = 3\sigma_{t,3}$. The expected payoff from adoption is $3\sigma_{t,3}v - c$, so a player is adopts in period $t + 1$ if $v > \frac{c}{3\sigma_{t,3}}$. Since v is uniform on $[0, 1]$, the best response map is then

$$\sigma_{t+1,3} = 1 - \frac{c}{3\sigma_{t,3}}.$$

To compute the steady states, we solve for fixed points of the best response map. We solve

$$\sigma = 1 - \frac{c}{3\sigma} \implies 3\sigma^2 - 3\sigma + c = 0 \implies \sigma = \frac{3 \pm \sqrt{9 - 12c}}{6}.$$

Positive steady state adoption levels are possible if $c \leq 3/4$.

For c strictly below $\frac{3}{4}$, there are two equilibria. Notice that in between the two steady states at $\sigma_{t,3} = \frac{1}{2}$, the best response is $1 - \frac{c}{3/2} = 1 - \frac{2c}{3} > \frac{1}{2}$, where the inequality follows because $c < \frac{3}{4}$. Best response dynamics lead to increasing adoption for $\sigma_{t,3}$ between the steady states, and decreasing adoption above the high equilibrium or below the low equilibrium. This means the high equilibrium is stable, and the low equilibrium is a tipping point. The non-adoption equilibrium is also stable.

- (b) Suppose now that D takes the values 2 and 4 with equal probability. Write down the neighbor degree distribution \tilde{D} . Compute the best response maps $\sigma_{t+1,2}$ and $\sigma_{t+1,4}$ as a function of $\sigma_{t,2}$ and $\sigma_{t,4}$. Find the steady state adoption levels.

From the definition we have

$$\mathbb{P}(\tilde{D} = 2) = \frac{\frac{1}{2} \cdot 2}{\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 4} = \frac{1}{3}.$$

We then have $\mathbb{P}(\tilde{D} = 4) = \frac{2}{3}$.

This means that for a degree 2 player with value v , the expected payoff to adoption is

$$2v \left(\frac{1}{3}\sigma_{t,2} + \frac{2}{3}\sigma_{t,4} \right) - c,$$

and for a degree 4 player it is

$$4v \left(\frac{1}{3}\sigma_{t,2} + \frac{2}{3}\sigma_{t,4} \right) - c.$$

Define $\sigma_t = \frac{1}{3}\sigma_{t,2} + \frac{2}{3}\sigma_{t,4}$. A degree 2 player adopts if $v > \frac{c}{2\sigma_t}$ and a degree 4 player adopts if $v > \frac{c}{4\sigma_t}$. The best response maps are then

$$\sigma_{t+1,2} = 1 - \frac{c}{2\sigma_t}, \quad \sigma_{t+1,4} = \frac{c}{4\sigma_t}.$$

To find steady state adoption levels, we can compute

$$\sigma_{t+1} = \frac{1}{3}\sigma_{t+1,2} + \frac{2}{3}\sigma_{t+1,4} = 1 - \frac{c}{3\sigma_t}$$

and solve for the fixed points. Note the fixed points are exactly the same as in part (a). Given the fixed point σ , a fraction

$$\sigma_2 = 1 - \frac{c}{2\sigma}$$

of degree 2 players adopt, and a fraction

$$\sigma_4 = 1 - \frac{c}{4\sigma}$$

degree 4 players adopt.

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