Recursive Methods

Outline Today's Lecture

- Dynamic Programming under Uncertainty
  notation of sequence problem
- leave study of dynamics for next week
- Dynamic Recursive Games: Abreu-Pearce-Stachetti
- Application: today's Macro seminar

Dynamic Programming with Uncertainty

- general model of uncertainty: need Measure Theory
- for simplicity: finite state S
- Markov process for s (recursive uncertainty)

$$\Pr\left(s_{t+1}|s^t\right) = p\left(s_{t+1}|s_t\right)$$

$v^*\left(x_0, s_0\right) \equiv$

$$\sup_{\{x_{t+1}(\cdot)\}_{t=0}^{\infty}} \left\{\sum_t \sum_{s^t} \beta^t F\left(x_t\left(s^{t-1}\right), x_{t+1}\left(s^t\right)\right) \Pr\left(s^t|s_0\right)\right\}$$

$$x_{t+1}\left(s^t\right) \in \Gamma\left(x_t\left(s^{t-1}\right)\right)$$

x_0 given

Dynamic Programming

Functional Equation (Bellman Equation)

$$v(x,s) = \sup \left\{ F(x,y) + \beta \sum_{s'} v(y,s')\, p(s'|s) \right\}$$

or simply (or more generally):

$$v(x,s) = \sup \{ F(x,y) + \beta E[v(y,s')\,|s] \}$$

where the $E[\cdot|s]$ is the conditional expectation operator over s' given s

- basically same: Ppple of Optimality, Contraction Mapping (bounded case), Monotonicity [actually: differentiability sometimes easier!]
- notational gain is huge!

Policy Rules Rule

- more intuitive too!
- fundamental change in the notion of a solution

optimal policy $g(x,s)$
vs.
optimal sequence of contingent plan $\{x_{t+1}(s^t)\}_{t=0}^{\infty}$

- Question: how can we use g to understand the dynamics of the solution? (important for many models)
- Answer: next week...

Abreu Pearce and Stachetti (APS)

- Dynamic Programming for Dynamic Games
- **idea**: subgame perfect equilibria of repeated games have recursive structure

  $\rightarrow$ players care about future strategies only through their associated utility values
- APS study general N person game with non-observable actions
- we follow Ljungqvist-Sargent:

  continuum of identical agents vs. benevolent government
- time consistency problems (credibility through reputation)
- agent i has preferences $u(x_i, x, y)$ where x is average across x_i's

One Period

- competitive equilibria:

$$C = \left\{ (x, y) : x \in \arg\max_{x_i} u(x_i, x, y) \right\}$$

assume $x = h(y)$ for all $(x, y) \in C$

1. Dictatorial allocation: $\max_{x,y} u(x, x, y)$ (wishful thinking!)
2. Ramsey commitment allocation: $\max_{(x,y)\in C} u(x, x, y)$ (wishful thinking?)
3. Nash equilibrium (x^N, y^N): (might be **bad outcome**)

$$\begin{aligned} x^N &\in \arg\max_{x} u(x, x^N, y^N) \Leftrightarrow (x^N, y^N) \in C \\ y^N &\in \arg\max_{y} u(x^N, x^N, y) \Leftrightarrow y^N = H(x^N) \end{aligned}$$

Kydland-Prescott / Barro-Gordon

$$
\begin{aligned}
v(u,\pi) &= -u^2 - \pi^2 \\
u &= \bar{u} - (\pi - \pi^e)
\end{aligned}
$$

$$
\begin{aligned}
u(\pi_i^e, \pi^e, \pi) &= v(\bar{u} - (\pi - \pi^e), \pi) - \lambda (\pi_i^e - \pi)^2 \\
&= -(\bar{u} - (\pi - \pi^e))^2 - \pi^2 - \lambda (\pi_i^e - \pi)^2
\end{aligned}
$$

then $\pi_i^e = \pi^e = \pi = h(\pi)$ take $\lambda \to 0$ then

$$
-(\bar{u} - \pi + \pi^e)^2 - \pi^2
$$

- First best Ramsey:

$$
\begin{aligned}
\max_{\pi} \left\{ -(\bar{u} - \pi + h(\pi))^2 - \pi^2 \right\} &= \max_{\pi} \left\{ -(\bar{u})^2 - \pi^2 \right\} \\
&\rightarrow \pi^* = 0
\end{aligned}
$$

Kydland-Prescott / Barro-Gordon

- Nash outcome. Gov't optimal reaction:

$$\max_{\pi} \left\{ -\left(\bar{u} - \pi + \pi^e\right)^2 - \pi^2 \right\}$$

$$\pi = \frac{\bar{u} + \pi^e}{2}$$

  this is $\pi = H\left(\pi^e\right)$

- Nash equilibria is then $\pi = H\left(h\left(\pi\right)\right) = H\left(\pi\right) = \frac{\bar{u}+\pi}{2}$ which implies

$$\pi^{eN} = \pi^N = \bar{u}$$

  $\rightarrow$ unemployment stays at $\bar{u}$ but positive inflation $\Rightarrow$worse off

- Andy Atkeson: adds shock θ that is private info of gov't (macro seminar)

Infinitely Repeated Economy

- Payoff for government:

$$V_g = \frac{1-\delta}{\delta} \sum_{t=1}^{\infty} \delta^t r\left(x_t, y_t\right)$$

  where $r\left(x, y\right) = u\left(x, x, y\right)$

- strategies σ...

$$\begin{aligned} \sigma_g &= \left\{\sigma_t^g\left(x^{t-1}, y^{t-1}\right)\right\}_{t=0}^{\infty} \\ \sigma_h &= \left\{\sigma_t^h\left(x^{t-1}, y^{t-1}\right)\right\}_{t=0}^{\infty} \end{aligned}$$

- induce $\{x_t, y_t\}$ from which we can write $V_g\left(\sigma\right)$.
- continuation stategies: after history $\left(x^t, y^t\right)$ we write $\sigma|_{\left(x^t, y^t\right)}$

Subgame Perfect Equilibrium

- A strategy profile $\sigma = (\sigma^h, \sigma^g)$ is a *subgame perfect equilibrium* of the infinitely repeated economy if for each $t \geq 1$ and each history $(x^{t-1}, y^{t-1}) \in X^{t-1} \times Y^{t-1}$,

    1. The outcome $x_t = \sigma_t^h(x^{t-1}, y^{t-1})$ is a competitive equilibrium given that $y_t = \sigma_t^g(x^{t-1}, y^{t-1})$, i.e. $(x_t, y_t) \in C$
    2. For each $\hat{y} \in Y$

    $$(1-\delta)r(x_t, y_t)+\delta V_g(\sigma|_{(x^t, y^t)}) \geq (1-\delta)r(x_t, \hat{y})+\delta V_g(\sigma|_{(x^t; y^t-1, \hat{y})})$$

    (one shot deviations are not optimal)

Lemma

Take σ and let x and y be the associated first period outcome. Then σ is sub-game perfect if and only if:

1. for all $(\hat{x}, \hat{y}) \in X \times Y$ $\sigma|_{\hat{x},\hat{y}}$ is a sub-game perfect equilibrium
2. $(x, y) \in C$
3. $\hat{y} \in Y$

$$(1-\delta) r(x_t, y_t) + \delta V_g(\sigma|_{(x,y)}) \geq (1-\delta) r(x_t, \hat{y}) + \delta V_g(\sigma|_{(\hat{x},\hat{y})})$$

- note the stellar role of $V_g(\sigma|_{(x,y)})$ and $V_g(\sigma|_{(\hat{x},\hat{y})})$, its all that matters for checking whether it is best to do x or deviate...
- idea! think about values as fundamental

Values of all SPE

- Set V of values

$$V = V_g(\sigma)|\sigma \text{is a subgame perfect equilibrium}$$

- Let $W \subset R$. A 4-tuple $(x, y, \omega_1, \omega_2)$ is said to be *admissible with respect to W* if $(x, y) \in C$, $\omega_1, \omega_2 \in W \times W$ and

$$(1-\delta)r(x,y) + \delta\omega_1 \geq (1-\delta)r(x,\hat{y}) + \delta\omega_2,\ \forall \hat{y} \in Y$$

B(W) operator

Definition: For each set $W \subset R$, let $B(W)$ be the set of possible values $\omega = (1-\delta)r(x,y)+\delta\omega_1$ associated with some admissible tuples (x,y,ω_1,ω_2) wrt W :

$$B(W) \equiv \left\{ w : \begin{array}{l} \exists\,(x,y) \in C \text{ and } \omega_1,\omega_2 \in W \text{ s.t.} \\ (1-\delta)r(x,y)+\delta\omega_1 \geq (1-\delta)r(x,\hat{y})+\delta\omega_2,\ \forall \hat{y} \in Y \end{array} \right\}$$

- note that V is a fixed point $B\left(V\right) = V$
- we will see that V is the biggest fixed point

- **Monotonicity of B**. If $W \subset W' \subset R$ then $B(W) \subset B(W')$
- **Theorem (self-generation):** If $W \subset R$ is bounded and $W \subset B(W)$ (*self-generating*) then $B(W) \subset V$
- **Proof**
  - Step 1 : for any $W \in B(W)$ we can choose and x, y, ω_1, and ω_2

$$(1-\delta)r(x,y) + \delta\omega_1 \geq (1-\delta)r(x,\hat{y}) + \delta\omega_2,\ \forall \hat{y} \in Y$$

  - Step 2: for $\omega_1, \omega_2 \in W$ thus do the same thing for them as in step 1

    continue in this way...

Three facts and an Algorithm

- $V \subset B(V)$
- If $W \subset B(W)$, then $B(W) \subset V$ (by self-generation)
- B is monotone and maps compact sets into compact sets
- **Algorithm:** start with W_0 such that $V \subset B(W_0) \subset W_0$ then define $W_n = B^n(W_0)$

$$W_n \to V$$

Proof: since W_n are decreasing (and compact) they must converge, the limit must be a fixed point, but V is biggest fixed point

Finding V

In this simple case here we can do more...

- lowest v is self-enforcing

  highest v is self-rewarding

$$v_{low} = \min_{\substack{(x,y)\in C \\ v\in V}} \left\{(1-\delta)\, r\,(x,y) + \delta v\right\}$$

$$(1-\delta)r(x,y) + \delta v \geq (1-\delta)r(x,\hat{y}) + \delta v_{low} \text{ all } \hat{y} \in Y$$

$$\Rightarrow v_{low} = (1-\delta)r(h\,(y)\,, y) + \delta v \geq (1-\delta)r(h\,(y)\,, H\,(h\,(y))) + \delta v_{low}$$

- if binds and $v > v_{low}$ then minimize RHS of inequality

$$v_{low} = \min_{y} r(h\,(y)\,, H\,(h\,(y)))$$

Best Value

- for Best, use Worst to punish and Best as reward

  solve:

  $$\max_{\substack{(x,y)\in C \\ v\in V}} = \{(1-\delta)\, r\,(x,y) + \delta v_{high}\}$$

  $$(1-\delta)r(x,y) + \delta v_{high} \geq (1-\delta)r(x,\hat{y}) + \delta v_{low} \text{ all } \hat{y} \in Y$$

  then clearly $v_{high} = r\,(x,y)$

- so

  $$\max r\,(h\,(y)\,, y)$$

  subject to $r\,(h\,(y)\,, y) \geq (1-\delta)r(h\,(y)\,, H\,(h\,(y))) + \delta v_{low}$

- if constraint not binding $\rightarrow$ Ramsey (first best)

- otherwise value is constrained by v_{low}