14.123 Microeconomic Theory III. 2014

Problem Set 2. Solution.

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1. In all counter-examples, I use $s = \frac{1}{3}$, $c = 1$ and suppose that whenever indifferent between n and n', Ann chooses cruder partition. In counter-examples, acts are constant on $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ and so, it is optimal for Ann to choose only between $n = 0, 1$. I also use notation $\mathbb{E}[u(f(s)) : B] = \int\limits_B u(f(s))ds$ and $\mathbb{E}_n[u(f(s)) : B] = \int\limits_{B\cap(\frac{1}{4},\frac{1}{2})} u(f(s))ds$

   1.1 Completeness holds. Given n the preferences are complete. Since $u(f(s))$ is bounded, without loss of generality, I can restrict the choice of n to some finite set and so, optimal n exists.

   1.2 Transitivity fails. Consider $f(s) = \begin{cases} -1 & s < \frac{1}{2}, \\ 1 & s \geq \frac{1}{2}; \end{cases}$, $h(s) = \begin{cases} 2 & s < \frac{1}{2}, \\ -2 & s \geq \frac{1}{2}; \end{cases}$, $g(s) = 0$. Then $f \sim_s g$ (for $n = 0$, both give expected utility 0; for $n = 1$, Ann's expected utility is $\frac{1}{2} < c$ and so, $n = 0$ is optimal)

   and $g \sim_s h$ (for $n = 0$, both give expected utility 0; for $n = 1$, Ann's expected utility is $\frac{1}{2}0 + \frac{1}{2}2 = 1 = c$ and so, $n = 0$ is optimal),

   but $h \succ_s f$ (for $n = 0$, both give expected utility 0; for $n = 1$, Ann's expected utility is $\frac{1}{2}1 + \frac{1}{2}2 = \frac{3}{2} > c$ and so, $n = 1$ is optimal).

   1.3 P2 holds. Let f, f', g, g' be defined as in P2 in the lecture notes for some $B \subset S$. Let n and n' be optimal levels of contemplation for comparison of f and g, and f' and g', respectively. Observe that $n = n'$, since

$$\mathbb{E}_n u(f(s)) - \mathbb{E}_n u(g(s)) = \mathbb{E}_n[u(f(s)) : B] - \mathbb{E}_n[u(g(s)) : B] =$$

$$= \mathbb{E}_n[u(f'(s)) : B] - \mathbb{E}_n[u(g'(s)) : B] = \mathbb{E}_n u(f'(s)) - \mathbb{E}_n u(g'(s)).$$

   Then

$$f \succeq_s g \iff \mathbb{E}_n u(f(s)) \geq \mathbb{E}_n u(g(s)) \iff \mathbb{E}_n[u(f(s)) : B] \geq \mathbb{E}_n[u(g(s)) : B] \iff$$

$$\iff \mathbb{E}_n[u(f'(s)) : B] \geq \mathbb{E}_n[u(g'(s)) : B] \iff \mathbb{E}_n u(f'(s)) \geq \mathbb{E}_n u(g'(s)) \iff f' \succeq_s g'.$$

1.4 P3 holds. Consider x, x' and $f = x^h_{|B}$, $f' = x'^h_{|B}$ for some act $h \in F$ and $B \subset S$. Observe that $n = 0$ is optimal for comparison of f and f', as f is (weakly) better than f' state by state. Therefore, $f \succ_s f' \iff \mathbb{E}u(f(s)) > \mathbb{E}u(f'(s)) \iff x \succ x'$, since B is non-null.

1.5 P4 fails. Consider $A = [0, \frac{1}{2}]$, $B = (\frac{1}{2}, 1]$, and $x = 1, x' = -1, y = 2, y' = -2$. Then $f_A \sim_s f_B$ (for $n = 0$, both give expected utility 0; for $n = 1$, Ann's expected utility is 1, but she needs to incur contemplation costs $c = 1$ and so, $n = 0$ is optimal),

but $g_A \succ_s g_B$ (as before, for $n = 0$, both give expected utility 0; for $n = 1$ Ann's expected utility is 2 which after subtracting contemplation costs gives payoff 1 and so, $n = 1$ is optimal).

1.6 P5 holds by the assumption that Z contains at least two elements.

2. I am looking for a concave utility function $u \in \mathcal{U}$ that satisfies $\frac{1}{2}u(\omega_0+G)+\frac{1}{2}u(\omega_0-L) = u(\omega_0)$ and $.6u(\omega_0+1)+.4u(\omega_0-1) = u(\omega_0)$ with the smallest reward G. To find such G, I need to find $u \in \mathcal{U}$ such that the utility gain from G is as big as possible, and the utility loss from L is as small as possible. Given the restriction to concave functions, it's clear that the optimal u should be linear on intervals where it is not specified by constraints on u and should match the derivatives at the boundaries of intervals.

2.1 Here, u is only specified at three points $(\omega_0, u(\omega_0)), (\omega_0 + 1, u(\omega_0 + 1)), (\omega_0 - 1, u(\omega_0-1))$, so we do linear extrapolation on the rest of the domain. Therefore,

$$\frac{u(\omega_0 + G) - u(\omega_0)}{u(\omega_0) - u(\omega_0 - L)} = \frac{2G}{3L},$$

and at the same time

$$\frac{u(\omega_0 + G) - u(\omega_0)}{u(\omega_0) - u(\omega_0 - L)} = 1,$$

so $G = 150000$.

2.2 Here, u is specified only on $[\omega_0-100, \omega_0+100]$. From $.6e^{-\alpha(\omega_0+1)}+.4e^{-\alpha(\omega_0-1)} = e^{-\alpha\omega_0}$, find $\alpha = \ln 1.5$ and so, $u'(\omega_0 + 100) = \ln 1.5(1.5)^{-(\omega_0+100)}$ and $u'(\omega_0 - 100) = \ln 1.5(1.5)^{-(\omega_0-100)}$. By the linearity of u outside $[\omega_0 - 100, \omega_0 + 100]$,

$$u(\omega_0 + 100 + (G - 100)) = u(\omega_0 + 100) + u'(\omega_0 + 100)(G - 100),$$

$$u(\omega_0 - 100 - (L - 100)) = u(\omega_0 - 100) - u'(\omega_0 - 100)(L - 100).$$

Now using the second indifference condition, we get

$$G = 100 + \frac{-2 + 1.5^{100} + 1.5^{-100} + \ln 1.5(1.5)^{100}(L - 100)}{\ln 1.5(1.5)^{-100}}.$$

2.3 From the previous part $\alpha = \ln 1.5$ and so, $\left(\frac{2}{3}\right)^G + \left(\frac{2}{3}\right)^{-L} = 2$ which is not possible.

2.4 Let $x = \omega_0^{-1}$. By CRRA specification,

$$.6(1 + x)^{1-\rho} + .4(1 - x)^{1-\rho} = 1 = .5(1 + Gx)^{1-\rho} + .5(1 - Lx)^{1-\rho}.$$

By $\omega_0 \geq L \gg 0$, we have $x \ll 0$ and so, we could take the Taylor expansion of the first equation to get

$$.6(1 - \rho)x + .6(1 - \rho)\rho\frac{x^2}{2} = .4(1 - \rho)x - .4(1 - \rho)\rho\frac{x^2}{2} + o(x^2)$$

or $\rho x = .4 + o(x^2)$ and so, $\rho \gg 1$. Now $(1 - Lx)^{1-\rho} \geq \exp(Lx(\rho - 1)) \approx \exp(.4L\frac{\rho-1}{\rho}) \gg 2$ and so, it is impossible to find appropriate G.