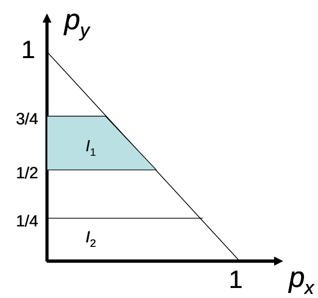
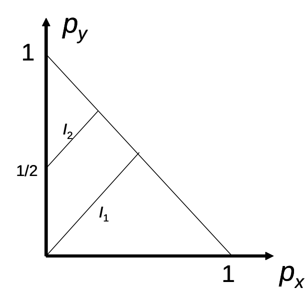
14.123 Microeconomic Theory III
Problem Set 1

1. Let P be the set of all lotteries $p = (p_x, p_y, p_z)$ on a set $C = \{x, y, z\}$ of consequences. Below, you are given pairs of indifference sets on P. For each pair, check whether the indifference sets belong to a preference relation that has a Von-Neumann and Morgenstern representation (i.e. expected utility representation). If the answer is Yes, provide a Von-Neumann and Morgenstern utility function; otherwise show which Von-Neumann and Morgenstern axiom is violated. (In the figures below, setting $p_z = 1 - p_x - p_y$, we describe P as a subset of $\mathbb{R}^2$.)

   (a) $I_1 = \{p|1/2 \le p_y \le 3/4\}$ and $I_2 = \{p|p_y = 1/4\}$:



   (b) $I_1 = \{p|p_y = p_x\}$ and $I_2 = \{p|p_y = p_x + 1/2\}$:



2. For any preference relation $\succeq$ that satisfies the Independence Axiom, show that the following are true.

   (a) For any $p, q, r, r' \in P$ with $r \sim r'$ and any $a \in (0, 1]$,

$$ap + (1-a)r \succeq aq + (1-a)r' \iff p \succeq q. \quad (1)$$

   (b) For any $p, q, r \in P$ and any real number a such that $ap+(1-a)\,r, aq+(1-a)r \in P$,

$$\text{if } p \sim q, \text{ then } ap + (1-a)\,r \sim aq + (1-a)r. \quad (2)$$

   (c) For any $p, q \in P$ with $p \succ q$ and any $a, b \in [0, 1]$ with $a > b$,

$$ap + (1-a)\,q \succ bp + (1-b)\,q. \quad (3)$$

(d) There exist $c^B, c^W \in C$ such that for any $p \in P$,

$$c^B \succeq p \succeq c^W. \tag{4}$$

[Hint: use completeness and transitivity to find $c^B, c^W \in C$ with $c^B \succeq c \succeq c^W$ for all $c \in C$; then use induction on the number of consequences and the Independence Axiom.]

3. Let P be the set of probability distributions on $C = \{x, y, z\}$. Find a continuous preference relation $\succeq$ on P, such that the indifference sets are all straight lines, but $\succeq$ does not have a von Neumann-Morgenstern utility representation.

4. Let $\dot{\succeq}$ be the "at least as likely as" relation defined between events in Lecture 3. Show that $\dot{\succeq}$ is a qualitative probability.