

14.12 Game Theory – Midterm I

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Instructions. This is a **closed book** exam. You have 90 minutes. You need to show your work when it is needed. All questions have equal weights. You may be able to receive partial credit for stating the *relevant* facts, such as the definition of the solution concept, towards the correct solution. Also, if you leave the answer for a part blank or just write "I don't know the answer", you will receive 10% of the full grade for that part. Good luck!

1. Consider the following game.

	w	x	y	z
a	3,3	2,1	0,2	2,1
b	1,1	1,2	1,0	1,4
c	0,0	1,0	3,2	1,1
d	0,0	0,5	0,2	3,1

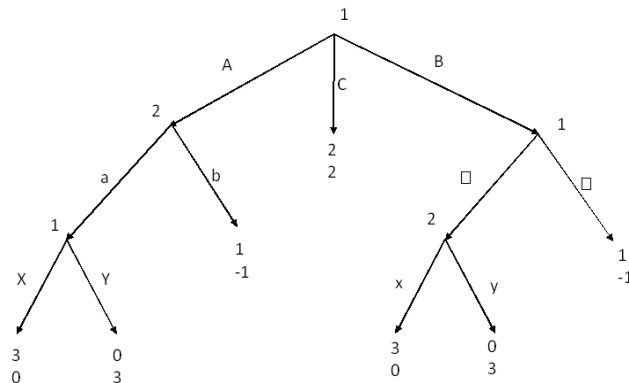
- (a) Compute the set of all rationalizable strategies.

Solution: b is strictly dominated by a mixed strategy that plays with a with probability $\frac{1}{2}$ and c with $\frac{1}{2}$. Hence, b is eliminated. Then, z is strictly dominated by y and eliminated. Next, d is strictly dominated by a mixed strategy that plays with a with probability $\frac{1}{2}$ and c with $\frac{1}{2}$ and eliminated. Lastly, x is strictly dominated by y and eliminated. For remaining strategies, we cannot eliminate any more as a is a BR (best response) to w , c is BR to y , w is BR to a , and y is BR to c . The set of all rationalizable strategies is $\{a, c\} \times \{w, y\}$.

- (b) Compute the set of all Nash equilibria.

Solution: Pure strategy Nash Equilibria are (a, w) and (c, y) . To find a mixed strategy NE, note that any such equilibrium puts positive probability only on rationalizable strategies. Hence, player 1 can mix between a and c , and player 2 can mix between w and y . Suppose player 2 plays w with probability p and y with probability $1 - p$. To make player 1 indifferent, $3p = 3(1 - p)$, so $p = \frac{1}{2}$. Similarly, if player 1 plays a with probability q and c with probability $1 - q$, to make player 2 indifferent, $3q = 2q + 2(1 - q)$ should hold. This gives $q = \frac{2}{3}$.

2. Consider the following extensive form game.



- (a) Apply backward induction to find an equilibrium.

Solution: We start from the bottom cases. For the bottom left node, player 1 will choose X and for the bottom right node, player 2 will choose y . Then, for the middle left node, player 2 will choose a and for the middle right node, player 1 will choose β . Lastly, at the top node, player 1 will choose A . Thus, NE is $(AX\beta, ay)$.

- (b) Write this game in normal form.

Solution:

	ax	ay	bx	by
$AX\alpha$	3,0	3,0	1,-1	1,-1
$AX\beta$	3,0	3,0	1,-1	1,-1
$AY\alpha$	0,3	0,3	1,-1	1,-1
$AY\beta$	0,3	0,3	1,-1	1,-1
$BX\alpha$	3,0	0,3	3,0	0,3
$BX\beta$	1,-1	1,-1	1,-1	1,-1
$BY\alpha$	3,0	0,3	3,0	0,3
$BY\beta$	1,-1	1,-1	1,-1	1,-1
$CX\alpha$	2,-2	2,-2	2,-2	2,-2
$CX\beta$	2,-2	2,-2	2,-2	2,-2
$CY\alpha$	2,-2	2,-2	2,-2	2,-2
$CY\beta$	2,-2	2,-2	2,-2	2,-2

- (c) Find a Nash equilibrium that leads to a different outcome than that of the solution in (a).

Solution: $(CX\alpha, by)$ is another NE, as player 2 has no incentive to deviate if player 1 plays C and player 1 also has no incentive to deviate since if he plays A then he gets 1 and plays B then he gets either 0 or 1. The outcome of $(CX\alpha, by)$ is player 1 plays C , while the outcome in part (a) was that Player 1 plays A , Player 2 plays a and finally Player 1 plays X .

3. In a pirate ship, $n \geq 2$ pirates are to determine the amount y of gunpowder for the ship as follows. Simultaneously, each pirate i submits a real number $s_i \geq 0$. The amount of gunpowder is determined to be

$$y = \min \{s_1, \dots, s_n\},$$

and each pirate i pays his share y/n of the cost. The payoff of a pirate i is

$$u_i(y) = \sqrt{y} - y/n.$$

Everything above is commonly known. (You will get 75% of the points if you solve this problem for $n = 2$.)

- (a) Write this formally as a normal-form game.

Solution: Players: $\{1, 2, \dots, n\}$. Strategies: each pirate i (call P_i) chooses $s_i \in [0, \infty)$. Payoffs: $U_i(s_1, \dots, s_n) = \sqrt{\min \{s_1, \dots, s_n\}} - \frac{\min \{s_1, \dots, s_n\}}{n}$.

- (b) Check whether there is a dominant-strategy equilibrium. If there is one, compute it and verify that it is indeed a dominant-strategy equilibrium. Otherwise, explain why there cannot be a dominant strategy equilibrium.

Solution: Note that u_i is a concave function maximized at $y = n^2/4$. By concavity, u_i increases in y for $y < n^2/4$ and decreases in y for $y > n^2/4$. The strategy profile $(n^2/4, \dots, n^2/4)$ is a dominant strategy equilibrium. To show this, take any player i and any $s_i < n^2/4$. For any s_{-i} , writing $y(s_{-i}) \equiv \min_{j=i} s_j$, we have

$$\begin{aligned} U_i(n^2/4, s_{-i}) &= u_i(y(s_{-i})) = U_i(s_i, s_{-i}) && \text{if } y(s_{-i}) \leq s_i \\ U_i(n^2/4, s_{-i}) &= u_i(\min\{y(s_{-i}), n^2/4\}) > u_i(s_i) = U_i(s_i, s_{-i}) && \text{if } y(s_{-i}) > s_i. \end{aligned}$$

Therefore, $n^2/4$ weakly dominates s_i . Similarly, $n^2/4$ weakly dominates any $s_i > n^2/4$ because for any s_{-i}

$$\begin{aligned} U_i(n^2/4, s_{-i}) &= u_i(y(s_{-i})) = U_i(s_i, s_{-i}) && \text{if } y(s_{-i}) \leq n^2/4 \\ U_i(n^2/4, s_{-i}) &= u_i(n^2/4) > u_i(\min\{y(s_{-i}), s_i\}) = U_i(s_i, s_{-i}) && \text{if } y(s_{-i}) > n^2/4. \end{aligned}$$

4. Apply backward induction to compute an equilibrium in the following game. Alice sues a large corporation (defendant) for damages. At date $2n + 1$, the judge will decide whether the defendant is guilty. If the judge decides that the defendant is guilty, then the defendant will be ordered to pay 1 to Alice; otherwise there will be no payment between the parties. (Here, the unit of money is million US dollars.) The probability that the judge decides guilty is $p \in (0, 1)$. Before the court date, Alice and the defendant can settle out of court, in which case they do not go to court. The settlement negotiation is as follows. In each date $t \in \{1, \dots, 2n\}$, one of them is to make a settlement offer s_t , and the other party is to decide whether to accept it. If the offer is accepted, the game ends and the defendant pays s_t to Alice. Alice makes the offers on odd dates $1, 3, \dots, 2n - 1$, and the defendant makes the offers on even dates $2, \dots, 2n$. Alice's payoff from receiving payment x is $x^{1/\alpha}$ for some $\alpha > 1$. She does not discount the payoffs and does not pay any cost for negotiation or for going to court. On the other hand, the defendant is *risk neutral* and it needs to pay a small fee $c > 0$ to the lawyers for every day the case has not settled (paying ct if they settle at date t and $c(2n + 1)$ if they go to court). (You will get 50% of the points if you can solve it for $n = 1$.)

Solution: If they cannot settle, at date $2n + 1$, judge orders the defendant to pay 1 to Alice with probability p , and hence the expected payoffs of Alice and the defendant are

$$\begin{aligned} V_{A,2n+1} &= p1^{1/\alpha} + (1-p)0^{1/\alpha} = p \\ V_{D,2n+1} &= -p - (2n+1)c, \end{aligned}$$

respectively. Now at date $2n$, Alice must accept an offer s if and only if

$$s^{1/\alpha} \geq V_{A,2n+1} = p,$$

i.e., $s \geq p^\alpha$. Hence, the payoff of the defendant from an offer s is

$$U_{D,2n}(s) = \begin{cases} -s - 2nc & \text{if } s \geq p^\alpha, \\ -p - (2n+1)c & \text{otherwise.} \end{cases}$$

Since $p^\alpha < p + c$, $U_{D, n}(s)$ is maximized at $s = p^\alpha$, and the defendant offers

$$s_n = p^\alpha.$$

Consequently, at date $2n - 1$, the defendant accepts an offer s iff $s \leq s_n + c$. Since Alice will get only s_n for sure if the offer is rejected, she offers

$$s_{n-} = s_n + c.$$

For any date $t = 2k < 2n$, suppose that if the parties do not settle at t or before they will settle for s_{k+1} at $2k + 1$ (as above). Then, at t , Alice accepts an offer s iff $s \geq s_k$ where

$$s_k = s_{k+1},$$

and the defendant offers s_k . At date $2k - 1$, the defendant accepts an offer s iff $s \leq s_{k-}$, and Alice offers s_{k-} where

$$s_{k-} = s_k + c.$$

Note that the solution to the above difference equations is

$$s_k = s_{k+1} = p^\alpha + (n - k)c. \tag{1}$$

The resulting equilibrium is: at any even date t , Alice accepts an offer s iff $s \geq s_t$, and the defendant offers s_t ; at any odd date t , Alice offers s_t and the defendant accepts an offer s iff $s \leq s_t$, where s_t is as in (1). [You will lose 5 points if you get the last part wrong and reveal that you don't know the definition of a strategy or an equilibrium.]

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