

14.12 Game Theory
MIDTERM 1 SOLUTIONS
 10/16/2008

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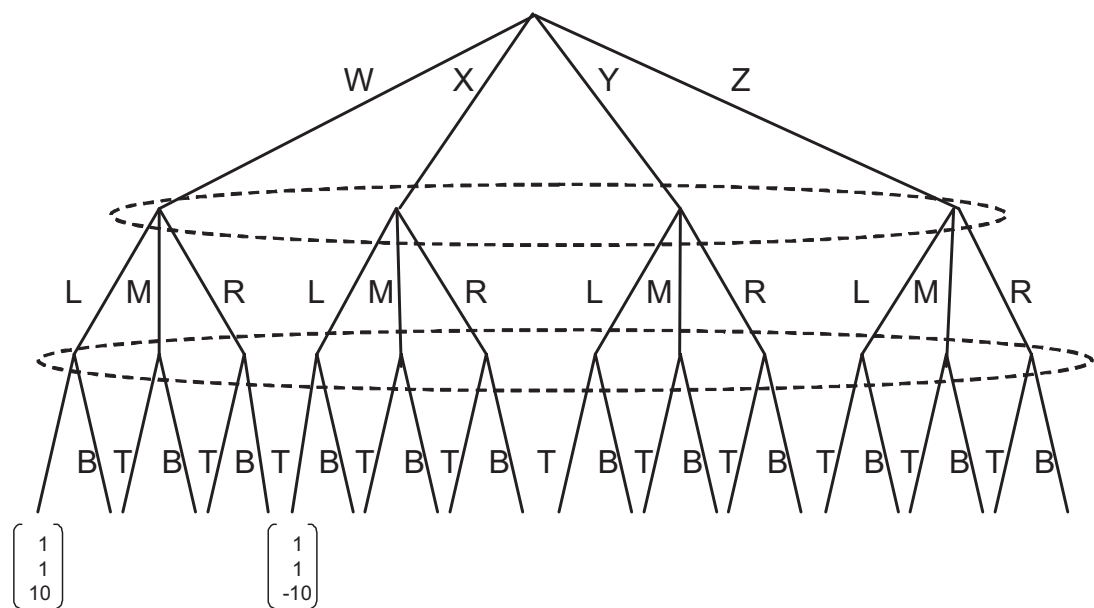
Instructions. This is an open book exam; you can use any written material. You may use a calculator. You may not use a computer or any electronic device with wireless communication capacity. You have one hour and 20 minutes. Each question is 25 points, and the breakdown of points within each question is specified below. Good luck!

1. A normal form Game is depicted below. Player 1 chooses the row (T or B), Player 2 chooses the column (L,M, or R), and Player 3 chooses the matrix (W,X,Y, or Z).

		W				X		
		L	M	R		L	M	R
T		1,1,10	1,0,10	5,-5,0	T	1,1,-10	1,0,10	5,-5,0
B		0,1,10	0,0,-10	5,-5,0	B	0,1,10	0,0,10	5,-5,0
		Y				Z		
		L	M	R		L	M	R
T		20,15,2	12,20,2	5,-5,40	T	8,8,-20	8,8,-20	5,40,-20
B		16,20,2	20,15,2	5,-5,40	B	8,8,-20	8,8,-20	5,40,-20

- (a) (5 points) Write a strategic form game tree for this game, and indicate the payoffs on any two terminal nodes of your choice. You *don't* need to write the payoffs at any other terminal nodes.

Answer. The strategic form game tree for this game is the following



(b) (5 points) Find utilities that player 1 can get by playing each of his actions are:

$$u_T = 20p + 12(1 - p) = 12 + 8p$$

$$u_B = 16p + 20(1 - p) = 20 - 4p$$

Player 1 will therefore be indifferent between his two actions if and only if

$$\begin{aligned} u_T &= u_B \\ 12 + 8p &= 20 - 4p \end{aligned}$$

which implies

$$p = \frac{2}{3}$$

Now, let's assume that player 1 plays T and B with probability α and $1 - \alpha$, respectively; then player 2 will get expected utilities given by

$$u_L = 15\alpha + 20(1 - \alpha) = 20 - 5\alpha$$

$$u_M = 20\alpha + 15(1 - \alpha) = 15 + 5\alpha$$

Therefore, player 2 will be indifferent and willing to randomize if and only if

$$\begin{aligned} u_L &= u_M \\ 20 - 5\alpha &= 15 + 5\alpha \end{aligned}$$

which implies

$$\alpha = \frac{1}{2}$$

Thus, if player 3 is forced to play Y , then the only mixed strategy Nash equilibrium

is given by

$$\begin{aligned} p(T) &= \frac{1}{2} \\ p(B) &= \frac{1}{2} \\ p(L) &= \frac{2}{3} \\ p(M) &= \frac{1}{3} \end{aligned}$$

- (c) (10 points) Find all of the rationalizable strategies in the full **3 player** game. Show your reasoning.

Answer. First of all, remember that, as we saw in problem set 1, in a three player game the set of rationalizable strategies corresponds to the set that survives iterated elimination of strictly dominated strategies only if we allow for the possibility of correlated beliefs. Therefore, if you simply eliminate iteratively all the strictly dominated strategies without specifying that you are assuming correlated beliefs, you lose some points. Alternatively, you can leave out correlated beliefs and check that all the strategies that survive iterated elimination are indeed best response to some rationalizable strategy.

First notice that strategy Z is strictly dominated for player 3. Once we eliminate Z , then R for player 2 becomes strictly dominated. There are no other strictly dominated strategies and the algorithm stops here. Thus, if you explicitly allow for correlated beliefs, the solution is (T, B) for player 1, (L, M) for player 2 and (W, X, Y) for player 3. However, if you don't mention specific assumptions about beliefs, then you need to check that each of the remaining strategies is a best response to some rationalizable strategy. In particular, notice that Y is not an optimal response to any rationalizable strategy. To see this, let α and p be the probabilities that T and L are played, respectively. Then player 3 will get utilities

$$\begin{aligned} u_W &= 10\alpha p + 10\alpha(1-p) + 10p(1-\alpha) - 10(1-\alpha)(1-p) \\ &= 20\alpha + 20p - 20p\alpha - 10 \\ u_X &= -10\alpha p + 10\alpha(1-p) + 10p(1-\alpha) + 10(1-\alpha)(1-p) \\ &= 10 - 20\alpha p \\ u_Y &= 2 \end{aligned}$$

Now, consider, for example, the mixed strategy $p(W) = p$ and $p(X) = 1-p$. This gives player 3 utility

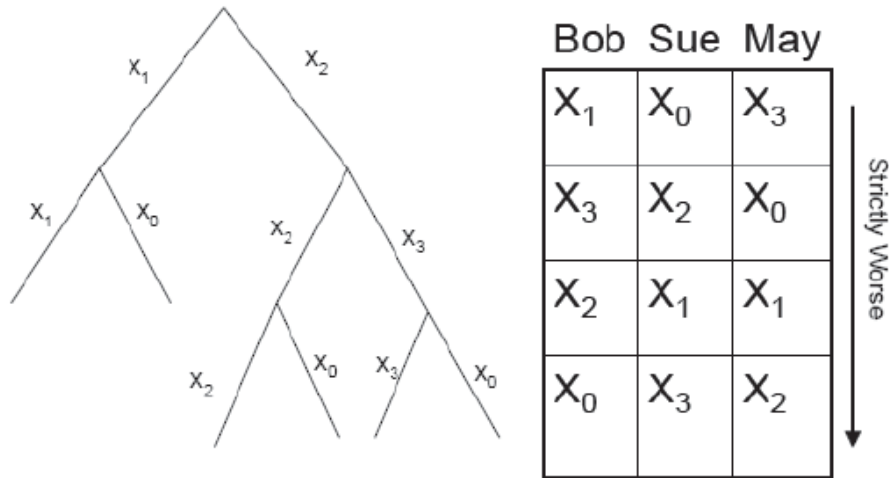
$$\begin{aligned} pu_W + (1-p)u_X &= 20\alpha p + 20p^2 - 20p^2\alpha - 10p + 10(1-p) - 20\alpha p(1-p) \\ &= 20\alpha p + 20p^2 - 20p^2\alpha - 10p + 10 - 10p - 20\alpha p + 20\alpha p^2 \\ &= 20p^2 - 20p + 10 \end{aligned}$$

which is always greater than $u_Y = 2$. Therefore, for each set of beliefs, player 3 can play a mixed strategy that gives him a payoff higher than Y . This means that

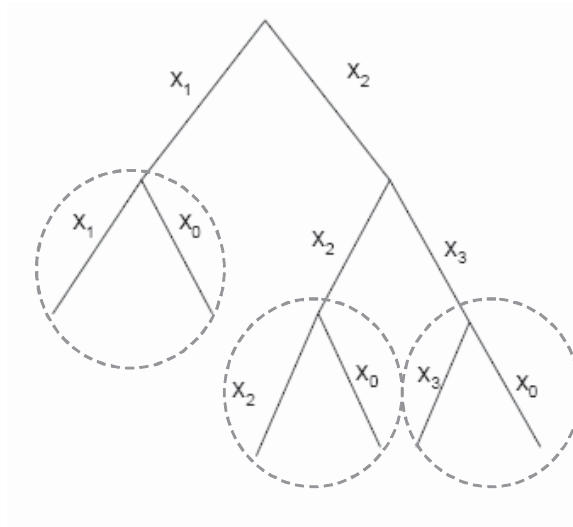
Y is never a best response. Once we eliminate Y , then strategy B for player 1 will be strictly dominated by T . The elimination of B makes strategy M strictly dominated for player 2. Finally, after eliminating M , strategy X will be strictly dominated for player 3. The only rationalizable strategies are therefore T for player 1, L for player 2 and W for player 3. Notice that these strategies form the unique Nash equilibrium found in part b.

2. “Quickies” Part (a) Required. CHOOSE 1 or (b) or (c).

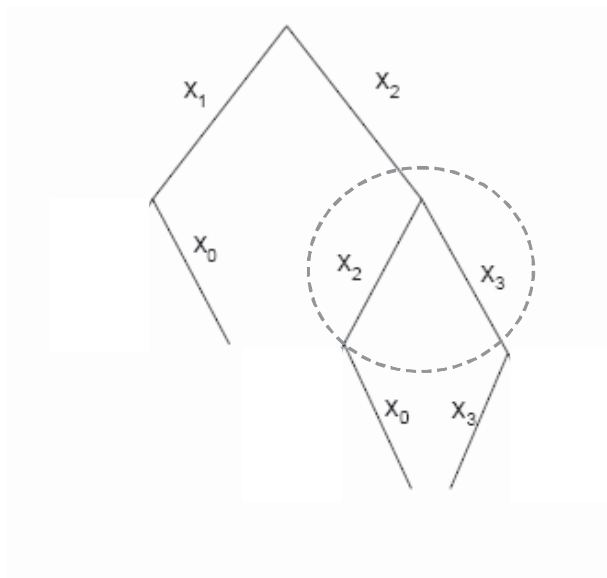
- (a) (REQUIRED; 15 points) If Bob, Sue and May are rational voters with strict preferences given in the table to the right, with top being better, and all this is common knowledge, what outcome do you expect the binary agenda at left to produce?



Answer. By assumption, Bob, Sue and May are *rational* voters. This means that, when they vote at a particular node, they take into account what will happen in the following nodes. Therefore, we can solve the game by *backward induction*. Let’s start from the penultimate nodes which are circled in the following picture:



When asked to vote between x_1 and x_0 the voters will choose x_0 (Sue and May prefer x_0 over x_1). In the same way, x_0 will be preferred over x_2 and x_3 over x_0 . Now, consider the node circled in the following graph:

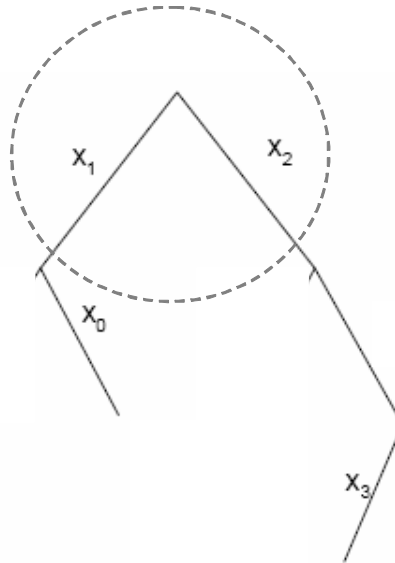


where I have eliminated the branches that will not be chosen. If voters are rational then they will understand that, if they choose x_2 , the final outcome will end up being x_0 , while if they opt for x_3 , then the final outcome will indeed be x_3 . In the node circled in the figure, therefore, the voters will vote for x_3 (Bob and May prefer this choice over x_0). Finally, let's consider the initial node:

Rational voters will understand that in the initial node they are not asked to choose between x_1 and x_2 , but instead between x_0 and x_3 . Therefore, they will choose x_2 and the final outcome of the voting agenda will be x_3 .

(b) **(CHOOSE (b) OR (c);10 points)** What, if anything, is wrong with the following pattern of choices? (If you don't have a calculator and want to know: $5/6 = 0.\bar{8}\bar{3}$.)

- Choice 1: $0.5[\$100] + 0.5[\$0] = p \succ q = 0.6[\$80] + 0.4[\$0]$.



- Choice 2: $1[\$80] = r \succ s = (5/6)[\$100] + (1/6)[\$0]$. (10 points)

Answer. Notice that lottery p can be rewritten as $0.6[s] + 0.4[\$0]$, while q is equivalent to $0.6[r] + 0.4[\$0]$. When transformed in this way, it is very easy to see that the two choices violate the independence axiom. The reason is that, if the independence axiom is true, then Choice 1 implies that $s \succ r$ which is the opposite of Choice 2.

(c) (If you already answered (b) don't do this – we won't grade it!) Consider a Judicial Settlement problem:

- At each date $t = 1, 2, \dots, n$ the **Plaintiff** makes a settlement offer s_t . The **Defendent** can either accept or reject each offer. (Note that the same player is making offers each period.)
- If the Defendent accepts at date t , the “game” ends with the Defendent paying s_t to the Plaintiff, and the Defendent and Plaintiff paying tc_D and tc_P to their respective lawyers.
- If the Defendent rejects at all dates, the case goes to court. The Defendent will lose and have to pay J to the Plaintiff. The Plaintiff and Defendent will also have to pay lawyer's fees $(n + 1)c_P$ and $(n + 1)c_D$ respectively.

If it is common knowledge that Plaintiff and Defendent are sequentially rational, how much will the settlement be, and at what date will it take place? (You **don't** have to show the backward induction reasoning explicitly. Just give the answer and 1 or two sentences of intuition.)

Answer. Under the assumption of common knowledge of sequential rationality of both players, we can solve the game by backward induction. In the penultimate period, $t = n$, the Plaintiff will offer a settlement offer s_n which makes the defendent exactly indifferent between accepting the offer and rejecting it and

paying $J + c_D$ in the court. In fact, any offer smaller than $s_n = J + c_D$ will result in a loss for the Plaintiff given that the Defendant will accept any offer smaller or equal to $J + c_D$. In the third to last period, $t = n - 1$, the Plaintiff will again make the Defendant indifferent between accepting and not accepting, that is, he will offer a settlement of $s_{n-1} = J + 2c_D$. Proceeding in this way, we find that in the initial node of the game, at time $t = 1$, the Plaintiff will make a settlement offer $s_1 = J + nc_D$ and the game will end with the Defendant accepting the offer.

3. In this question you are asked to compute the rationalizable strategies in a linear Bertrand-duopoly with discrete prices and fixed “startup” costs. We consider a world where the prices must be an odd multiple of 10 cents, i.e.,

$$P = \{0.1, 0.3, 0.5, \dots, 0.1 + 0.2n, \dots\}$$

is the set of feasible prices. For each price p , the demand is:

$$Q(p) = \max\{1 - p, 0\}.$$

We have two firms $N = \{1, 2\}$, each with 0 marginal cost, but each with a fixed “start-up” cost k . That is, if the firm produces a positive amount, it must bear the cost k . *If it produces 0, it does not have to pay k .* Simultaneously, each firm sets a price $p_i \in P$. Observing prices p_1 and p_2 , consumers buy from the firm with the lowest price. When prices are equal, they divide the demand equally between the two firms. Each firm i wishes to maximize its profit.

$$\pi_i(p_1, p_2) = \begin{cases} p_i Q(p_i) - k & \text{if } p_i < p_j \text{ and } Q(p_i) > 0 \\ p_i Q(p_i)/2 - k & \text{if } p_i = p_j \text{ and } Q(p_i) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) If $k = 0.1$:

- (5 points) Show that $p_i = 0.1$ is strictly dominated.

SOLUTION: Claim: $p_i = 0.1$ is strictly dominated by $p_i = 0.5$. (It is also strictly dominated by every other strategy EXCEPT for $p_i = 0.9$). Let us see this. Suppose $p_i = 0.1$ is played. The payoffs will be:

$$\pi_i(0.1, p_j) = \begin{cases} -0.055 & \text{when } p_j = 0.1 \\ -0.01 & \text{when } p_j > 0.1 \end{cases}$$

The payoff, as can be seen, will always be strictly less than zero. Therefore, as long as there is some strategy that ensures a payoff of at least 0 in all cases, it will strictly dominate playing 0.01. $p_i = 0.5$ is one of these strategies (but certainly not the only one). If 0.5 is the strictly smaller than the other players strategy, it will yield a payoff of 0.15, and if the other player also plays 0.5, the firm will get a payoff of 0.025. Finally, if the other player plays strictly less than 0.05, then the firm will get zero. Thus, it strictly dominates 0.1. (3 points credit given for showing that the payoff is negative and saying that this is strictly dominated by “not participating”. To get full credit, you must have pointed to a strategy (which was a price p) that strictly dominated 0.1 and explained why).

- (5 points) Show that there are prices greater than the monopoly price ($p = 0.5$) that are *not* strictly dominated.

SOLUTION: Suppose that the other firm plays 0.1. Then, you do NOT want to “win” the price war, since playing 0.1 will yield a negative payoff. Playing anything above 0.1 will be a best response as anything strictly above will yield a payoff of zero. Thus playing any strategy above the monopoly price will be a best response to the belief that the other firm will be playing 0.1. (They are weakly dominated, but NOT strictly dominated).

3. (15 points) Iteratively eliminate all strictly dominated strategies to find the set of rationalizable strategies. Explain your reasoning.

SOLUTION:

- First round: Eliminate 0.1 from both players strategies.
We know that we can eliminate 0.1 because it is strictly dominated by 0.5 as discussed in part (i).
- Second round: Eliminate all $p_i \geq 0.9$ from both players strategies. (All are strictly dominated by 0.3)
Look at $p_i = 0.3$. Note that:

$$\pi_i(0.3, p_j) = \begin{cases} 0.005 & \text{when } p_j = 0.3 \\ 0.11 & \text{when } p_j > 0.3 \end{cases}$$

So, once 0.1 is eliminated, 0.3 will always produce a payoff that is strictly greater than zero. Now consider the payoff of playing $p_i = 0.9$:

$$\pi_i(0.9, p_j) = \begin{cases} 0 & \text{when } p_j < 0.9 \\ -0.055 & \text{when } p_j = 0.9 \\ -0.01 & \text{when } p_j > 0.9 \end{cases}$$

Thus, playing 0.9 will never produce a payoff greater than zero, so it is strictly dominated by 0.3.. Playing $p_i > 0.9$ will always yield a payoff of exactly zero, so will also be strictly dominated by 0.3 which always yields positive payoff. (once 0.1 has been eliminated)..

- Third round: Eliminate $p_i = 0.7$ for both players as it is strictly dominated by 0.3.

Now we are just left with the strategies 0.3, 0.5, 0.7. Claim: playing 0.3 will strictly dominate playing 0.7. Let us look first at playing 0.3:

$$\pi_i(0.3, p_j) = \begin{cases} 0.005 & \text{when } p_j = 0.3 \\ 0.11 & \text{when } p_j = 0.5 \\ 0.11 & \text{when } p_j = 0.7 \end{cases}$$

Now look at playing 0.7 :

$$\pi_i(0.7, p_j) = \begin{cases} 0 & \text{when } p_j = 0.3 \\ 0 & \text{when } p_j = 0.5 \\ 0.005 & \text{when } p_j = 0.7 \end{cases}$$

Thus, it is clear that playing 0.3 strictly dominates playing 0.7. This leaves 0.3 and 0.5.

- Fourth round: Eliminate $p_i = 0.5$ for both players as is strictly dominated by 0.3.

If the other player plays 0.3, you get 0.005 for playing 0.3 and zero for playing 0.5, and if the other player plays 0.5, you get 0.11 for playing 0.3 and only 0.025 for playing 0.5). Thus 0.3 is the unique rationalizable strategy achieved through the iteration of strictly dominated strategies.

(Only partial credit given if the eliminations were made without saying what strategies were actually strictly dominating the strategies that were eliminated. For example, a number of people said that 0.9 could be eliminated in the first round. This is NOT true, as it is still a best response to 0.1.)

4. There are three “dates”, $t = 1, 2, 3$, and two players: Government and Worker.

- At $t = 1$, Worker expends effort to build $K \in [0, \infty)$ units of capital.
- At $t = 2$, Government sets tax rates $\tau_K \in [0, 1]$ and $\tau_e \in [0, 1]$ on capital-holdings and on labor income.
- At $t = 3$, Worker chooses effort $e_2 \in [0, \infty)$ to produce output Ke_2 .
The payoffs of Government and Worker are:

$$U_G = \tau_K K + \tau_e Ke_2$$

and

$$U_W = (1 - \tau_e)Ke_2 + (1 - \tau_K)K - K^2/2 - e_2^2/2.$$

(a) (20 points) Solve the game by backwards induction.

Solution:

- At $t = 3$, worker maximizes U_W taking everything else as given. The first order conditions give:

$$e_2^* = (1 - \tau_e)K.$$

(5 points)

- At $t = 2$, the Government chooses τ_K and τ_e to maximize U_G , recognizing that their choice of τ_e effects e_2^* . They thus maximize

$$\tau_K K + \tau_e K(1 - \tau_e)K$$

The first order condition for τ_e gives

$$\begin{aligned} \tau_e^* &= \frac{1}{2} \text{ if } K > 0 \\ \tau_e^* &\in [0, 1] \text{ if } K = 0. \end{aligned}$$

(5 points; -1 for not noting what happens if $K = 0$). The first order condition for τ_K doesn't have a solution: $\frac{\partial U_G}{\partial \tau_K} > 0$ if $K > 0$. Hence,

$$\begin{aligned} \tau_k^* &= 1 \text{ if } K > 0 \\ \tau_k^* &\in [0, 1] \text{ if } K = 0. \end{aligned}$$

(5 points).

- At $t = 1$, the worker chooses K to maximize U_W , recognizing that his choice will effect the tax rates and his future effort e_2 . Plugging in the solutions from $t = 2, 3$ for $K > 0$ gives:

$$\begin{aligned} U_W &= (1 - \frac{1}{2})K(1 - \frac{1}{2})K + (1 - 1)K - K^2/2 - \left((1 - \frac{1}{2})K \right)^2 / 2 \\ &= \frac{1}{8}K^2 - \frac{1}{2}K^2 = -\frac{3}{8}K^2. \end{aligned}$$

Hence, the U_W is maximized at $K = 0$. (5 points.)

- (b) (5 points) Now suppose before the game is played, Government can “delegate” its job to an independent IRS Agent at period $t = 0$. At $t = 0$, the Government will offer a fraction $\beta_K \in [0, 1]$ of its capital tax revenue and a fraction $\beta_e \in [0, 1]$ of its labor tax revenue to the Agent. The Agent can either Accept or Reject. If the Agent Accepts, she will take the place of the Government in setting tax rates $\tau_K \in [0, 1]$ and $\tau_e \in [0, 1]$ at $t = 2$. If the Agent Rejects, the game proceeds as before. The Agent has payoff:

$$U_A = \left\{ \begin{array}{l} \beta_K \tau_K K + \beta_e \tau_e K e_2 - \varepsilon \text{ if accept} \\ 0 \text{ if reject.} \end{array} \right\},$$

where ε is a very small but positive “acceptance” cost. The Government’s payoff will be:

$$U_G = \left\{ \begin{array}{l} (1 - \beta_K) \tau_K K + (1 - \beta_e) \tau_e K e_2 \text{ if Agent Accepts} \\ \tau_K K + \tau_e K e_2 \text{ otherwise} \end{array} \right\}.$$

Assume that an Agent who accepts will choose the smallest tax rate(s) consistent with sequential rationality. Find an equilibrium of the game using backward induction, and briefly comment on it.

Solution:

- As in part (a), at $t = 3$,

$$e_2^* = (1 - \tau_e)K.$$

- At $t = 2$, the same solutions apply if the Agent has Rejected. If the Agent Accepted and $\beta_k > 0$ and $\beta_e > 0$. If the Agent accepted and $\beta_i = 0$ for $i = e$ or k the Agent doesn’t care what τ_i is; by assumption they choose a tax rate of 0. So if the Agent Accepts:

$$\begin{aligned} \tau_e^* &= \frac{1}{2} \text{ if } K > 0 \text{ and } \beta_e > 0 \\ \tau_e^* &= 0 \text{ if } K = 0 \text{ or } \beta_e = 0 \end{aligned}$$

$$\begin{aligned} \tau_k^* &= 1 \text{ if } K > 0 \text{ and } \beta_k > 0 \\ \tau_k^* &\in 0 \text{ if } K = 0 \text{ or } \beta_k = 0. \end{aligned}$$

- At $t = 1$, $K = 0$, as in part (a) if the Agent Rejects *or if the Agent accepts and both* $\beta_k > 0$ and $\beta_e > 0$.. If $\beta_k > 0$ and $\beta_e = 0$, then the worker maximizes

$$(1 - 0)K(1 - 0)K + (1 - 1)K - K^2/2 - ((1 - 0)K)^2/2 = 0.$$

So any K is equally good.

$$K^* \in [0, \infty) \text{ if } \beta_k > 0 \text{ and } \beta_e = 0$$

If $\beta_k = 0$ and $\beta_e > 0$, then the worker maximizes

$$\begin{aligned} U_W &= (1 - \frac{1}{2})K(1 - \frac{1}{2})K + (1 - 0)K - K^2/2 - \left((1 - \frac{1}{2})K \right)^2 / 2 \\ &= K - \frac{3}{8}K^2. \end{aligned}$$

The first order conditions give

$$K^* = \frac{4}{3} \text{ if } \beta_k = 0 \text{ and } \beta_e > 0.$$

- At $t = 0$, the Agent clearly Rejects if $\beta = 0$ and $\beta_e = 0$. If $\beta = 0$ and $\beta_e > 0$, his payoff will be:

$$\frac{\beta_e}{2} \frac{4}{3} (1 - 1/2) \frac{4}{3} - \varepsilon.$$

So the Agent will Accept if $\beta_e > \frac{9}{4}\varepsilon$, Reject if $\beta_e < \frac{9}{4}\varepsilon$ and be willing to do either if $\beta_e = \frac{9}{4}\varepsilon$. If $\beta > 0$ and $\beta_e = 0$, whether or not the Agent will accept depends on what he anticipates K will be (since the worker will be indifferent) Since we're just told to find *an* equilibrium, let's assume $K = 0$ in this case.

- At $t = 0$, the Government will get a payoff of 0 by offering $\beta = 0$ and $\beta_e = 0$ (as in (a)) Given the assumption above that the Worker will choose $K = 0$ if $\beta > 0$ and $\beta_e = 0$, the Government will get a 0 payoff by offering $\beta > 0$ and $\beta_e = 0$ (also as in (a)). If Government offers $\beta = 0$ and $\beta_e \geq \frac{9}{4}\varepsilon$ the Agent will Accept, the worker will work, and the Government will get a positive payoff, (so long as $\beta_e < 1$, which is OK to assume, since ε is small). It wants to choose the smallest agent share, so it picks $\beta_e = \frac{9}{4}\varepsilon$.

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Fall 2012

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