14.06 Lecture Notes
Intermediate Macroeconomics

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Chapter 4

Applications

4.1 Consumption Smoothing

4.1.1 The Intertemporal Budget

- For any given sequence of interest rates $\{R_t\}_{t=0}^{\infty}$, pick an arbitrary $q_0 > 0$ and define q_t recursively by

$$q_t = \frac{q_{t-1}}{1 + R_t},$$

that is,

$$q_t = \frac{q_0}{(1 + R_1)...(1 + R_t)}.$$

q_t represents the price of period$-t$ consumption relative to period-0 consumption.

- The buget in period t is given by

$$c_t + a_{t+1} \leq (1 + R_t)a_t + y_t$$

where a_t denotes assets and y_t denotes labor income.

- Multiplying the period-t budget by q_t and adding up over all t, we get
$$\sum_{t=0}^{T}\left[q_t c_t + q_t a_{t+1}\right] \leq \sum_{t=0}^{T}\left[q_t(1+R_t)a_t + q_t y_t\right].$$
Using the fact that $q_t(1+R_t) = q_{t-1}$, we have
$$\sum_{t=0}^{T} q_t(1+R_t)a_t = q_0(1+R_0)a_0 + \sum_{t=1}^{T} q_{t-1}a_t$$
so that the above reduces to
$$\sum_{t=0}^{T} q_t c_t + q_T a_{T+1} \leq q_0(1+R_0)a_0 + \sum_{t=1}^{T} q_t y_t$$

- Assuming either that the agent dies at finite time without leaving any bequests, in which case $a_{T+1} = 0$, or that the time is infinite, in which case we impose $q_T a_{T+1} \to 0$ as $T \to \infty$, we conclude that the intertemporal budget constraint is given by
$$\sum_{t=0}^{T} q_t c_t \leq q_0(1+R_0)a_0 + \sum_{t=1}^{T} q_t y_t,$$
where $T < \infty$ (finite horizon) or $T = \infty$ (infinite horiozon). The interpretation is simple: The present value of the consumption the agent enjoys from period 0 and on can not exceed the value of initial assets the agent has in period 0 plus the present value of the labor income the agent receives from period 0 and on.

- We can rewrite the intertemporal budget as
$$\sum_{t=0}^{T} q_t c_t \leq q_0 x_0$$
where
$$x_0 \equiv (1+R_0)a_0 + h_0,$$
$$h_0 \equiv \sum_{t=0}^{\infty} \frac{q_t}{q_0} y_t.$$

$(1 + R_0)a_0$ is the household's *financial wealth* as of period 0. h_0 is the present value of labor income as of period 0; we often call h_0 the household's *human wealth* as of period 0. The sum $x_0 \equiv (1 + R_0)a_0 + h_0$ represents the household's *effective wealth*.

- Note that the sequence of per-period budgets and the intertemporal budget constraint are equivalent. We can then write the household's consumption problem as follows

$$\max \sum_{t=0}^{T} \beta^t U(c_t)$$
$$s.t. \sum_{t=0}^{T} q_t c_t \leq q_0 x_0$$

- Note that the above is like a "static" consumption problem: Interpret c_t as different consumption goods and q_t as the price of these goods. This observation relates to the context of Arrow-Debreu markets that we discuss later.

4.1.2 Consumption Smoothing

- The Lagrangian for the household's problem is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(c_t) + \lambda \left[q_0 x_0 - \sum_{t=0}^{T} q_t c_t \right]$$

where λ is the shadow cost of resources for the consumer (that is, the Lagrange multiplier for the intertemporal budget constraint).

- The FOCs give

$$U'(c_0) = \lambda q_0$$

for period 0 and similarly

$$\beta^t U'(c_t) = \lambda q_t$$

for any period t.

- Suppose for a moment that the interest rate equals the discount rate in all periods:

$$R_t = \rho \equiv 1/\beta - 1.$$

Equivalently,

$$q_t = \beta^t q_0$$

The FOCs then reduce to

$$U'(c_t) = \lambda q_0$$

for all t, and therefore

$$c_t = \overline{c}.$$

for all t. That is, the level of consumption is the same in all periods.

- But how is the value of $\overline{c}$ determined? From the intertemporal budget, using $q_t = \beta^t q_0$ and $c_t = \overline{c}$, we infer

$$q_0 x_0 = \sum_{t=0}^{T} q_t c_t = \frac{1}{1-\beta} q_0 \overline{c}$$

and therefore

$$\overline{c} = (1-\beta)x_0 = (1-\beta)\left[(1+R_0)a_0 + h_0\right]$$

That is, the household consumes a fraction of his initial effective wealth. This fraction is given by $1-\beta$.

4.2 Arrow-Debreu Markets

4.2.1 Arrow-Debreu versus Radner

- We now introduce uncertainty...

- Let $q(s^t)$ be the period-0 price of a unite of the consumable in period t and event s^t and $w(s^t)$ the period-t wage rate in terms of period-t consumables for a given event s^t. $q(s^t)w(s^t)$ is then the period-t and event-s^t wage rate in terms of period-0 consunambles. We can then write household's consumption problem as follows

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) U\left(c^j(s^t), z^j(s^t)\right)$$
$$s.t. \sum_t \sum_{s^t} q(s^t) \cdot c^j(s^t) \leq q_0 \cdot x_0^j$$

where

$$x_0^j \equiv (1+R_0)a_0 + h_0^j,$$
$$h_0^j \equiv \sum_{t=0}^{\infty} \frac{q(s^t)w(s^t)}{q_0}[l^j(s^t) - T^j(s^t)].$$

$(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0. $T^j(s^t)$ is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. h_0^j is the present value of labor income as of period 0 net of taxes; we often call h_0^j the household's *human wealth* as of period 0. The sum $x_0^j \equiv (1+R_0)a_0^j + h_0^j$ represents the household's *effective wealth*.

4.2.2 The Consumption Problem with CEIS

- Suppose for a moment that preferences are separable between consumption and leisure and are homothetic with respect to consumption:

$$\begin{aligned} U(c,z) &= u(c) + v(z). \\ u(c) &= \frac{c^{1-1/\theta}}{1-1/\theta} \end{aligned}$$

- Letting μ be the Lagrange multiplier for the intertemporal budget constraint, the FOCs imply

$$\beta^t \pi(s^t) u'\left(c^j(s^t)\right) = \mu q(s^t)$$

for all $t \geq 0$. Evaluating this at $t = 0$, we infer $\mu = u'(c_0^j)$. It follows that

$$\frac{q(s^t)}{q_0} = \frac{\beta^t \pi(s^t) u'\left(c^j(s^t)\right)}{u'(c_0^j)} = \beta^t \pi(s^t) \left(\frac{c^j(s^t)}{c_0^j}\right)^{-1/\theta}.$$

That is, the price of a consumable in period t relative to period 0 equals the marginal rate of intertemporal substitution between 0 and t.

- Solving $q_t/q_0 = \beta^t \pi(s^t) \left[c^j(s^t)/c_0^j\right]^{-1/\theta}$ for $c^j(s^t)$ gives

$$c^j(s^t) = c_0^j \left[\beta^t \pi(s^t)\right]^\theta \left[\frac{q(s^t)}{q_0}\right]^{-\theta}.$$

It follows that the present value of consumption is given by

$$\sum_t \sum_{s^t} q(s^t) c^j(s^t) = q_0^{-\theta} c_0^j \sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta q(s^t)^{1-\theta}$$

Substituting into the resource constraint, and solving for c_0, we conclude

$$c_0^j = m_0 \cdot x_0^j$$

where

$$m_0 \equiv \frac{1}{\sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^\theta \left[q(s^t)/q_0\right]^{1-\theta}}.$$

Consumption is thus linear in effective wealth. m_0 represent the MPC out of effective wealth as of period 0.

4.2.3 Intertemporal Consumption Smoothing, with No Uncertainty

- Consider for a moment the case that there is no uncertainty, so that $c^j(s^t) = c_t^j$ and $q(s^t) = q_t$ for all s^t.

- Then, the riskless bond and the Arrow securities satisfy the following arbitrage condition
$$q_t = \frac{q_0}{(1+R_0)(1+R_1)...(1+R_t)}.$$
Alternatively,
$$q_t = q_0 \left[1 + \widetilde{R}_{0,t}\right]^{-t}$$
where $\widetilde{R}_{0,t}$ represents the "average" interest rate between 0 and t. Next, note that m_0 is decreasing (increasing) in q_t if and only if $\theta > 1$ ($\theta < 1$). It follows that the marginal propensity to save in period 0, which is simply $1 - m_0$, is decreasing (increasing) in $\widetilde{R}_{0,t}$, for any $t \geq 0$, if and only if $\theta > 1$ ($\theta < 1$).

- A similar result applies for all $t \geq 0$. We conclude

Proposition 22 *Suppose preferences are seperable between consumption and leisure and homothetic in consumption (CEIS). Then, the optimal consumption is linear in contemporaneous effective wealth:*
$$c_t^j = m_t \cdot x_t^j$$
where
$$x_t^j \equiv (1+R_t)a_t^j + h_t^j,$$
$$h_t^j \equiv \sum_{\tau=t}^{\infty} \frac{q_t}{q_t}[w_\tau l_\tau^j - T_\tau^j],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$

m_t is a decreasing (increasing) function of q_τ for any $\tau \geq t$ if and only $\theta > 1$ ($\theta < 1$). That is, the marginal propensity to save out of effective wealth is increasing (decreasing) in future interest rates if and only if the elasticity of intertemporal substitution is higher (lower) than unit. Moreover, for given prices, the optimal consumption path is independent of the timining of either labor income or taxes.

- Obviously, a similar result holds with uncertainty, as long as there are complete Arrow-Debreu markets.

- Note that any expected change in income has no effect on consumption as long as it does not affect the present value of labor income. Also, if there is an innovation (unexpected change) in income, consumption will increase today and for ever by an amount proportional to the innovation in the annuity value of labor income.

- To see this more clearly, suppose that the interest rate is constant and equal to the discount rate: $R_t = R = 1/\beta - 1$ for all t. Then, the marginal propensity to consume is

  $$m = 1 - \beta^\theta (1+R)^{1-\theta} = 1 - \beta,$$

  the consumption rule in period 0 becomes

  $$c_0^j = m \cdot \left[(1+R)a_0 + h_0^j\right]$$

  and the Euler condition reduces to

  $$c_t^j = c_0^j$$

  Therefore, the consumer choose a totally flat consumption path, no matter what is the time variation in labor income. And any unexpected change in consumption leads to a parallel shift in the path of consumption by an amount equal to the annuity value of the change in labor income. This is the manifestation of *intertemporal consumption smoothing.*

- More generally, if the interest rate is higher (lower) than the discount rate, the path of consumption is smooth but has a positive (negative) trend. To see this, note that the Euler condition is

$$\log c_{t+1} \approx \theta(R-\rho) + \log c_t.$$

4.2.4 Incomplete Markets and Self-Insurance

- The above analysis has assumed no uncertainty, or that markets are complete. Extending the model to introduce idiosyncratic uncertainty in labor income would imply an Euler condition of the form

$$u'(c_t^j) = \beta(1+R)\mathbb{E}_t u'(c_{t+1}^j)$$

Note that, because of the convexity of u', as long as $Var_t[c_{t+1}^j] > 0$, we have $\mathbb{E}_t u'(c_{t+1}^j) > u'(\mathbb{E}_t c_{t+1}^j)$ and therefore

$$\frac{\mathbb{E}_t c_{t+1}^j}{c_t^j} > [\beta(1+R)]^\theta$$

This extra kick in consumption growth reflects the *precautionary motive for savings.* It remains true that transitory innovations in income result to persistent changes in consumption (because of consumption smoothing). At the same time, consumers find it optimal to accumulate a *buffer stock,* as a vehicle for self-insurance.