

Midterm Exam Solutions

Part A (True, false or uncertain)

1. *An economy that increases its saving rate will experience faster growth.*

Uncertain. In the Solow model an economy that increases its saving rate will temporarily experience faster growth, but the long-run growth rate remains unchanged. In endogenous growth models the long-run growth rate may be an increasing function of the saving rate. An example for this is the *AK* model.

2. *Once human capital is included, the Solow model does a good job at explaining observed per capita income differences across countries.*

Uncertain. Introducing human capital has the potential to greatly increase the ability of the Solow model to account for cross-country differences in per capita income. To see that, recall that in class we derived that in the Solow model, the long run elasticity of output with respect to the saving rate is $\frac{\alpha}{1-\alpha}$, where α is capital's share. If capital's share is moderate, this elasticity is not large. So differences in the savings rate across countries do not translate into large differences in per capita income across countries. But if capital's share is close to one, a small increase in s causes a large increase in per capital output. Recognizing the existence of human capital implies that we must raise our estimate of the share of income that is paid to capital (the usual estimate we used was one third), and this larger capital's share may enable us to explain large differences in per capita income across countries.

However, according to empirical studies by Hall and Jones (1999) and Klenow and Rodriguez-Clare (1997), variations in both physical and human capital are of non-negligible importance, but variations in output for given capital stocks are the most important source of income differences across countries (see Romer, Part B of Chapter 3).

3. *Growth theory predicts that poor countries should grow faster than rich countries, and researchers have found the evidence to be consistent with this prediction.*

The hypothesis that poor economies tend to grow faster per capita than rich ones – without conditioning on any other characteristics of economies – is referred to as *absolute convergence*. Does the Solow model predict absolute convergence? If all countries have the same parameters (technology, saving rate, etc.) then they all have the same balanced growth path and only differ by how far they are away from

the balanced growth path, and indeed the model would predict absolute convergence. If countries have differ in technology, savings rate etc., then the model does not necessarily predict absolute convergence. But one may still expect absolute convergence to occur. If only saving rates differ, then countries with low capital per worker due to a low savings rate will have a higher marginal product of capital, which provides incentives of capital to flow from rich to poor countries. If differences in per capita income are due to differences in technology, one may expect these differences to shrink as poorer countries gain access to state-of-the art methods.

Is the evidence consistent with absolute convergence? In class we discussed the study of Baumol (1986), and his results provided strong support for absolute convergence. However, De Long (1988) demonstrates that Baumol's findings are largely spurious due to problems of sample selection and measurement error (see Romer, section 1.7).

4. *A Ramsey economy that starts above the golden rule capital stock will converge to the golden rule. But if it starts below the golden rule it will also converge to a level below the golden rule because impatient consumers are unwilling to sacrifice consumption during the transition.*

False. The economy will converge to the modified golden rule capital stock irrespective of whether the initial stock is below or above the golden rule capital stock. Recall the Euler equation of the Ramsey model:

$$\frac{\dot{c}_t}{c_t} = \sigma(c_t)[f'(k_t) - \theta - n]$$

Only the modified golden rule level k^* implicitly defined by $f'(k^*) = n + \theta$ is consistent with zero consumption growth. If $f'(k_t) > \theta + n$, then consuming the same today and tomorrow is not optimal because then the the marginal rate of transformation would exceed the marginal rate of substitution and the representative family could increase utility by increasing consumption tomorrow relative to today. An analogous argument applies to the case $f'(k_t) < \theta + n$.

Part B

1. This specification can be interpreted as a learning-by-doing model in which knowledge accumulation occurs as a side effect of goods production. The idea of learning-by-doing is that, as individuals produce goods, they inevitably think of ways of improving the production process. For example, Arrow (1962) cites the empirical regularity that after a new airplane design is introduced, the time required to build the marginal aircraft is decreasing in the number of aircraft of that model that have already been produced. In contrast to R&D activities, this type of knowledge accumulation is not a result of deliberate efforts, but a side effect of conventional economic activity. (see Romer, page 120)
2. This question turned out to be an unintended trick question. If the specification had been $A(t) = Y(t)^\phi$, then $\phi = 0$ would have implied $A(t) = 1$ for all t , and the model would have reduced to the Solow model without exogenous technological progress. We added this question after writing and solving the problem, thinking that it would give you the opportunity to get some points without a lot of algebra, and we thought $\phi = 0$ was the answer. But with the specification $\dot{A}(t) = Y(t)^\phi$, $\phi = 0$ implies $\dot{A}(t) = 1$ for all t , and so $A(t) = A(0) + t$ and $g_A(t) = \frac{\dot{A}(t)}{A(t)} = \frac{1}{A(0)+t}$. So there is always some technological progress, but this technological progress eventually becomes negligible. So in the long run the economy will look very similar to a Solow economy without technological progress, but the equations governing the economy are not identical.
3. There was a typo in the question which was corrected during the exam. The question should have read “*Find expressions for $g_A(t)$ and $g_K(t)$ in terms of $A(t)$, $K(t)$, $L(t)$ and the parameters.*”, and the answer is

$$g_A(t) = \frac{Y(t)^\phi}{A(t)} = K(t)^{\phi\alpha} A(t)^{\phi(1-\alpha)-1} L(t)^{\phi(1-\alpha)},$$
$$g_K(t) = s \left(\frac{A(t)L(t)}{K(t)} \right)^{1-\alpha}.$$

4. Differentiation yields

$$\frac{\dot{g}_A(t)}{g_A(t)} = \phi\alpha g_K(t) + (\phi(1-\alpha) - 1)g_A(t) + \phi(1-\alpha)n,$$
$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)(g_A(t) + n - g_K(t)).$$

You can get this answer very quickly by using the following rules:

(a) If $X(t)$, $Y(t)$ and $Z(t)$ are functions of time and $Z(t) = X(t) \cdot Y(t)$, then

$$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) If $X(t)$ and $Y(t)$ are functions of time and $Y(t) = X(t)^\alpha$, then

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{X}(t)}{X(t)}.$$

You should have been aware of these rules from problem sets and recitation, but for some reason most of you did not apply them, and thus wasted precious time on tedious algebra.

What is important here is that the right hand side of the resulting equations depend on time only through $g_K(t)$ and $g_A(t)$, i.e. $K(t)$, $A(t)$ and $L(t)$ no longer appear in these equations. If this were not the case, it would not be possible to draw a phase diagram.

The $\dot{g}_A = 0$ and $\dot{g}_K = 0$ schedules are given by

$$g_K = \frac{1 - \phi(1 - \alpha)}{\phi\alpha} g_A - \frac{\phi(1 - \alpha)}{\phi\alpha} n, \quad (1)$$

$$g_K = g_A + n, \quad (2)$$

respectively. The two schedules intersect in $(g_A^*, g_K^*) = \left(\frac{\phi}{1-\phi}n, \frac{1}{1-\phi}n\right)$. The phase diagram is shown in figure 1.

5. As is clear from the phase diagram the economy converges to a balanced growth path on which A grows at rate g_A^* and both K and Y grow at rate g_K^* . The balanced growth path essentially looks like the balanced growth path of the Solow model, but instead of exogenous technological progress at rate g we now have technological progress at an endogenously determined rate $g_A^* = \frac{\phi}{1-\phi}n$.
6. What we are usually interested in is the growth rate of *per capita* output. Here the growth rate of output per capita $\frac{Y}{L}$ on the balanced growth path is given by $g_{\frac{Y}{L}}^* = g_A^* = \frac{\phi}{1-\phi}n$. An increase in s has no effect on long run per capita growth while an increase in n will increase long run per capita growth. The long run growth rate goes to infinity as ϕ gets close to one.
7. The growth rate of output per capita is not on average higher in countries with faster population growth, so this is not a good model to account for differences in growth rates across countries.

8. The private marginal product of capital as a function of $K(t)$, $L(t)$, $A(t)$ and the parameters is given by

$$r(t) = \alpha \left(\frac{A(t)L(t)}{K(t)} \right)^{1-\alpha}.$$

9. The Euler equation is

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\sigma}.$$

To derive it (not required in the exam), notice that the flow budget constraint of the representative household is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t),$$

so the Hamiltonian is

$$H(t) = e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} + \mu(t)[r(t)A(t) + w(t)L(t) - C(t)].$$

The two first order conditions used in the derivation of the Euler equation are $\frac{\partial H(t)}{\partial C(t)} = 0$, which can be written as

$$e^{-\rho t} C(t)^{-\sigma} = \mu(t)$$

and $\dot{\mu}(t) = -\frac{\partial H(t)}{\partial A(t)}$, which can be rewritten as

$$\frac{\dot{\mu}(t)}{\mu(t)} = -r(t).$$

Differentiating the first equation with respect to time yields

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\rho - \sigma \frac{\dot{C}(t)}{C(t)}.$$

Combining these results yields the Euler equation.

To give an economic interpretation of this condition, suppose it were violated in the following way:

$$\frac{\dot{C}(t)}{C(t)} < \frac{r(t) - \rho}{\sigma},$$

In this situation the interest rate (the marginal rate of transformation) exceeds the marginal rate of substitution between consumption tomorrow and consumption today, and the representative household can do better by consuming more tomorrow relative to today, i.e. by increasing consumption growth. An analogous argument rules out the reversed inequality. Here σ is the inverse of the elasticity of intertemporal substitution. If this elasticity is high, the household is very willing to substitute consumption across time, and as a consequence consumption growth is very responsive to differences between the interest rate and the rate of time preference.

10. This part is very difficult and actually nobody solved it. Here we are dealing with a three-dimensional dynamic system. The three variables I will use are g_A , g_K and r (there are more convenient choices, but this one is more consistent with the computations for a constant savings rate). It is straightforward to compute

$$\begin{aligned}\frac{\dot{g}_A(t)}{g_A(t)} &= \phi\alpha g_K(t) + (\phi(1-\alpha) - 1)g_A(t) + \phi(1-\alpha)n, \\ \frac{\dot{r}(t)}{r(t)} &= (1-\alpha)(g_A(t) + n - g_K(t)).\end{aligned}$$

The equation of motion for capital is $\dot{K}(t) = Y(t) - C(t)$, so

$$g_K(t) = \frac{r(t)}{\alpha} - \frac{C(t)}{K(t)}.$$

Differentiation yields

$$\begin{aligned}\dot{g}_K(t) &= \frac{1}{\alpha} \frac{\dot{r}(t)}{r(t)} r(t) - \left[\frac{\dot{C}(t)}{C(t)} - \frac{\dot{K}(t)}{K(t)} \right] \frac{C(t)}{K(t)} \\ &= \frac{(1-\alpha)}{\alpha} (g_A(t) + n - g_K(t)) r(t) - \left[\frac{r(t) - \rho}{\sigma} - g_K(t) \right] \left[\frac{r(t)}{\alpha} - g_K(t) \right].\end{aligned}$$

Then the growth rates on the balanced growth path are $g_C^* = g_Y^* = g_K^* = \frac{1}{1-\phi}n$ and $g_A^* = \frac{\phi}{1-\phi}n$, and the interest rate on the balanced growth path is $r^* = \frac{\sigma}{1-\phi}n + \rho$. The condition $\rho > \frac{1-\sigma}{1-\phi}n$ insures that $r^* > g_Y^*$, which is needed because otherwise the household could attain infinite lifetime utility.

Endogenizing savings does not change the growth rates on the balanced growth path. In particular, they do not depend on σ and ρ . This is not surprising given that the growth rates were also independent of the exogenous savings rate. Savings behavior, whether optimizing or given by a constant savings rate, does not affect long run growth in this model.

