14.05 Lecture Notes

The Neoclassical Growth Model

(aka The Ramsey Model)

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1 The Planner's Problem: Pareto Optimal Allocations

- In the Solow model, agents in the economy (and the planner) follow a simplistic linear rule for consumption and investment. In the Ramsey model, agents (and the planner) choose consumption and investment optimally so as to maximize their utility (welfare).

- In this section, we start the analysis of the neoclassical growth model by considering the optimal plan of a benevolent social planner, who chooses the static and intertemporal allocation of resources in the economy so as to maximize social welfare. We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

1.1 Preferences

- preferences are defined over streams of consumption, $\mathbf{c} = \{c_t\}_{t=0}^{\infty}$
- we assume that preferences can be represented by the following intertemporal utility function:

$$\mathcal{U}_t = \sum_{\tau=0}^{\infty} \beta^{\tau} U(c_{t+\tau})$$

- β is called the *discount factor*, with $\beta \in (0, 1)$.
- U is sometimes called the per-period utility or felicity function. we assume that U is strictly increasing and strictly concave, with $U'(0) = \infty$ and $U'(\infty) = 0$.
- usual specification with constant elasticity of intertemporal substitution (CEIS):

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta},$$

where $\theta > 0$ is the elasticity of intertemporal substitution.

1.2 Technology and the Resource Constraint

- We abstract from population growth and exogenous technological change.
- The resource constraint is given by

$$c_t + i_t \leq y_t$$

- Let $F(K, L)$ be a neoclassical technology and let $f(\kappa) = F(\kappa, 1)$ be the intensive form of F. Output in the economy is given by

$$y_t = F(k_t, 1) = f(k_t),$$

- Capital accumulates according to

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

(If there is positive population growth, re-interpret δ as the effective depreciation rate, $\delta + n$)

- Combining the above, we can rewrite the *resource constraint* as

$$c_t + k_{t+1} \leq f(k_t) + (1 - \delta)k_t$$

- Finally, we impose the following natural non-negativitly constraints:

$$c_t \geq 0, \quad k_t \geq 0.$$

1.3 The Planner's Problem (a.k.a. the Ramsey Problem)

- The social planner chooses a plan $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ so as to maximize utility subject to the resource constraint of the economy, taking initial k_0 as given. Formally,

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} \leq (1-\delta)k_t + f(k_t) \;\; \forall t \geq 0,$$

$$c_t \geq 0, \quad k_{t+1} \geq 0 \;\; \forall t \geq 0,$$

$$k_0 > 0 \;\; \textit{given.}$$

1.4 Solving the Planner's problem

- Let μ_t denote the Lagrange multiplier for the resource constraint. The Lagrangian of the social planner's problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \mu_t \left[(1-\delta)k_t + f(k_t) - k_{t+1} - c_t\right]$$

- Let $\lambda_t \equiv \beta^{-t}\mu_t$ be the multiplier in period-t terms. We can then rewrite the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{U(c_t) + \lambda_t \left[(1-\delta)k_t + f(k_t) - k_{t+1} - c_t\right]\right\}$$

- We henceforth assume an interior solution—as long as $k_t > 0$, interior solution is indeed ensured by the Inada conditions on F and U.

- The FOC with respect to c_t gives

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow U'(c_t) = \lambda_t$$

- The FOC with respect to k_{t+1}, on the other hand, gives the so-called Euler condition:

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta \left[1 - \delta + f'(k_{t+1})\right] \lambda_{t+1}$$

- Finally, the FOC with respect to λ_t simply gives us back the resource constraint:

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t)$$

- The Lagrange multiplier λ_t measures the marginal value of wealth (or resources) in period t: if we exogenously give the economy ϵ units of the good during period t, where ϵ is small enough, welfare increases by approximately $\lambda_t \epsilon$.

- Combining the above, we get the following alternative representation of the Euler condition:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + f'(k_{t+1}).$$

- This condition imposes equality between the marginal rate of intertemporal substitution in consumption and the corresponding marginal rate of transformation, which is simply the marginal capital of capital net of depreciation (plus one).

- Recall that we found a similar condition when we studied the optimal consumption/saving of a single individual. The only difference is that there the rate at which the individual could substitute consumption today for consumption tomorrow was given by the market interest rate, while here the rare at which the economy as a whole (the planner) can substitute consumption today for consumption tomorrow is given by the MPK (marginal product of capital) net of depreciation. This observation also anticipates that the socially optimal allocation can be replicated as a competitive market equilibrium in which the interest rate is equate with the net-of-depreciation MPK.

- Suppose for a moment that the horizon was finite, $T < \infty$. Then, the Lagrangian would be

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ U(c_t) + \lambda_t \left[(1-\delta)k_t + f(k_t) - k_{t+1} - c_t \right] \right\}$$

and the Kuhn-Tucker condition with respect to k_{T+1} would give

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} \geq 0 \quad \text{and} \quad k_{T+1} \geq 0, \text{ with complementary slackness;}$$

equivalently

$$\beta^T \lambda_T \geq 0 \;\text{ and }\; k_{T+1} \geq 0, \;\text{ with }\; \beta^T \lambda_T k_{T+1} = 0.$$

The latter means that either $k_{T+1} = 0$, or otherwise it better be that the shadow value of k_{T+1} is zero. When $T = \infty$, the terminal condition $\beta^T \lambda_T k_{T+1} = 0$ is replaced by the so-called transversality condition:

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0,$$

which means that the (discounted) shadow value of capital converges to zero. Equivalently,

$$\lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = 0.$$

- One can show that the aforementioned transversality condition is both necessary and sufficient (along with the Euler condition). Summarizing the preceding analysis, we thus reach the following.

Proposition 1 *The plan $\{c_t, k_t\}_{t=0}^{\infty}$ is a solution to the social planner's problem if and only if*

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + f'(k_{t+1}), \tag{1}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \tag{2}$$

for all $t \geq 0$,

$$k_0 > 0 \text{ given, and } \lim_{t\to\infty} \beta^t U_c'(c_t)k_{t+1} = 0. \tag{3}$$

- Note that this is essentially a system of two difference equations in two variables, c_t and k_t. In general, such a system admits multiple solutions: there are multiple paths of c_t and k_t that satisfy these two difference equations. Only one solution however satisfies the two relevant boundary conditions, which are initial condition for capital and the aforementioned transversality condition. This particular solution identifies the socially optimal plan.

- Also note that the above two equation have a very simple interpretation. The second condition is simply the resource constraint, which summarizes *feasibility*. The first condition, on the other hand, is the FOC in the planner's problem, which therefore summarizes *optimality*.

- This optimality condition gives, in effect, the optimal growth in consumption as a function of the marginal product of capital, the discount rate, and the elasticity of intertemporal substitution. To see this more clearly, suppose that preferences take the CEIS form,

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta}.$$

Then, the planner's optimality condition (1) reduces to

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+1}) - \delta]\}^{\theta}$$

It is then evident that the optimal consumption growth is positive ($c_{t+1}/c_t > 1$) if and only if the MPK net of depreciation exceeds the discount rate:

$$\frac{c_{t+1}}{c_t} > 1 \quad \Leftrightarrow \quad \beta[1 + f'(k_{t+1}) - \delta] > 1 \quad \Leftrightarrow \quad f'(k_{t+1}) - \delta > \rho$$

where, recall, $\beta \equiv \frac{1}{1+\rho}$. Furthermore, for any given positive gap between the MPK and the discount rate, consumption growth is higher the higher the elasticity of intertemporal substitution, θ. This is for the same reasons as in the consumer's problem we had analyzed before, except that now you should think of the planner substituting consumption today versus tomorrow.

- Turning now attention to the boundary conditions, note that capital is historically predetermined (k is a "state" variable). This immediately implies that the one boundary condition is simply the exogenously given initial level of capital, $k_0 = \bar{k}_0$.

- By contrast, consumption is not historically predetermined (c is a "control" or "jump" variable). Hence, the initial value of consumption, c_0, is *not* exogenously given. Instead, c_0 must be chose optimal so that the dynamics of consumption of capital satisfy an appropriate terminal condition.

- If the time horizon had been finite, this terminal condition would simply be $k_{T+1} = 0$, with T denoting the last period of life; that is, the planner would make sure that the economy eats all its capital before it dies. Now that the horizon is infinite, it is no more optimal to drive capital to zero at any finite time. Instead, the appropriate terminal condition becomes the "transversality" condition, namely

$$\lim_{t\to\infty} \beta^t U_c'(c_t) k_{t+1} = 0.$$

2 Decentralized Competitive Equilibrium

- As discussed in class, the allocation chosen by the planner coincides with the unique equilibrium allocation. This is merely an application of the two welfare theorems.

- Thus, the preceding proposition also characterizes the equilibrium of the economy.

- To see this more clearly, consider first the representative household. This has the same preferences as the planner but faces a different constraint: the individual budget constraint has to with market opportunities rather than technological constraints.

- Thus consider a market structure where the typical household supplies its labor in a competitive labor market for a wage rate w_t, can accumulate physical capital and rent it out to firms for a rental rate r_t, and can finally trade bonds for a (real) interest rate R_t.

- The budget constraint is then give by

$$c_t + k_{t+1} + b_{t+1} = (1 - \delta)k_t + r_t k_t + (1 + R_t)b_t + w_t \tag{4}$$

- We can thus write the representative household's problem as follows:

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} + b_{t+1} = (1 - \delta + r_t)k_t + (1 + R_t)b_t + w_t \quad \forall t \geq 0,$$

$$c_t \geq 0, \quad k_{t+1} \geq 0 \ \ \forall t \geq 0,$$

$$k_0 > 0 \ \ \textit{given}.$$

- Following a similar procedure like the one we used to solve the planner's problem, we can show that, in any interior allocation, the intertemporal optimality conditions for capital accumulation is given by

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + r_{t+1}$$

and similarly the one for bond savings is given by

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + R_{t+1}$$

- It follows that, in equilibrium the return to bonds (the interest rate) and the return to capital must be equal:

$$R_t = r_t - \delta$$

  If this were not true, the household would have tried to arbitrage the difference, leading either to an infinite demand for bonds and an infinite accumulation of capital (in the case that $R_t < r_t - \delta$) or no accumulation of capital (in the case that $R_t > r_t - \delta$). Hence, for an equilibrium to exist with positive and finite capital, it'd better be that $R_t = r_t - \delta$.

- In what follows, we therefore impose $R_t = r_t - \delta$ and concentrate on the remaining optimality condition, which is

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + r_{t+1} \quad (= 1 + R_{t+1}) \tag{5}$$

  This condition, along with the budget constraint and the appropriate boundary conditions, determine the optimal consumption and saving plan of the household for any given sequence of the wage rate and the rental rate of capital.

- Turning to the firm's optimal behavior, note that this is exactly the same as in the Solow model: the firm takes the wage w_t and the rental rate r_t as given and seeks to maximize its profit. From the analysis we did before, we thus know that the wage must satisfy

$$w_t = F_L(k_t, 1) = f(k_t) - f'(k_t)k_t,$$

while the rental rate of capital must satisfy

$$r_t = F_K(k_t, 1) = f'(k_t)$$

And, by implication,

$$r_t k_t + w_t = f(k_t)$$

- Combining the last condition with the household's budget constraint (4), and imposing market clearing in the bond market (which means $b_t = 0$ since the net aggregate borrowing/lending for a closed economy is zero), we reach the following condition:

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t)$$

That is, we have shown that the equilibrium allocation satisfies the resource constraint of the economy.

- Next, combining the firm's optimality condition $r_t = f'(k_t)$ with the household's Euler condition (5), we reach the following condition:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + f'(k_{t+1})$$

  That is, we have shown that the equilibrium allocation satisfies the planner's Euler condition.

- Finally, the transversality condition for the household's problem is given by

$$\lim_{t \to \infty} \beta^t U'(c_t)(1 + r_t - \delta) = 0$$

  which together with $r_t = f'(k_t)$ gives us the planner's transversality condition.

- We have thus shown that the equilibrium allocation satisfies all the conditions that characterize planner's problem—and therefore that the equilibrium allocation coincides with the social optimal allocation, which is precisely a manifestation of the two fundamental welfare theorems.

3 Steady State

Proposition 2 *There exists a unique (positive) steady state* $(c^*, k^*) > 0$*. The steady-state values of the capital-labor ratio, the productivity of labor, the output-capital ratio, the saving rate, the wage rate, the rental rate of capital, and the interest rate are all independent of the utility function* U *and are pinned down uniquely by the technology* F*, the depreciation rate* δ*, and the discount rate* ρ*. In particular, the capital-labor ratio* k^* *equates the net-of-depreciation MPK with the discount rate,*

$$f'(k^*) - \delta = \rho,$$

and is a decreasing function of $\rho + \delta$*, where* $\rho \equiv 1/\beta - 1$*. Similarly,*

$$R^* = \rho, \qquad r^* = \rho + \delta, \qquad w^* = F_L(k^*, 1),$$
$$y^* = f(k^*), \qquad \frac{y^*}{k^*} = \phi(k^*), \qquad s^* = \frac{\delta k^*}{y^*} = \frac{\delta}{\phi(k^*)},$$

where $f(k) \equiv F(k, 1)$ *and* $\phi(k) \equiv f(k)/k$*.*

- *Proof.* The Euler condition in steady state reduces to

$$1 = \beta[1 - \delta + f'(k^*)]$$

or equivalently

$$f'(k^*) - \delta = \rho$$

where ρ is the discount rate (with $\beta = \frac{1}{1+\rho}$). That is, the steady-state capital-labor ratio is pinned down uniquely by the equation of the MPK, net of depreciation, with the discount rate. It follows that the gross rental rate of capital and the net interest rate are $r^* = \rho + \delta$ and $R^* = \rho$, while the wage rate is $w^* = f(k^*) - f'(k^*)$. The output-capital ratio are given by $\frac{y^*}{k^*} = \phi(k^*)$, where $\phi(k) \equiv f(k)/k$. By the resource constraint, the steady-state level of consumption is given by

$$c^* = f(k^*) - \delta k^*$$

Finally, net investment is zero, gross investment is δk^*, and saving rate is

$$s^* = \frac{\delta k^*}{y^*} = \frac{\delta}{\phi(k^*)}$$

- The comparative statics are trivial. For example, an increase in β leads to an increase in $k = K/L$, $y = Y/L$, and the saving rate $s = \delta K/Y$. We could thus reinterpret the exogenous differences in saving rates assumed in the Solow model as endogenous differences in saving rates originating in exogenous differences in preferences.

- Homework: consider the comparative statics with respect to exogenous productivity or a tax on capital income.

4 Transitional Dynamics

- We now suppose that preferences exhibit constant elasticity of intertemporal substitution:

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta},$$

where $\theta > 0$ is the elasticity of intertemporal substitution.

- The Euler condition then reduces to

$$\frac{c_{t+1}}{c_t} = (\beta(1 + R_{t+1}))^{\theta} = \left(\frac{1 + R_{t+1}}{1 + \rho}\right)^{\theta},$$

or equivalently

$$\ln c_{t+1} - \log c_t \approx \theta(R_{t+1} - \rho)$$

where $R_{t+1} = f'(k_{t+1}) - \delta$. Thus, the elasticity of intertemporal substitution, θ, controls the sensitivity of consumption growth to the rate of return to savings.

Proposition 3 *The equilibrium path* $\{c_t, k_t\}_{t=0}^{\infty}$ *is given by the unique solution to*

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+}) - \delta]\}^{\theta},$$
$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t,$$

for all t*, with initial condition* $k_0 > 0$ *given and terminal condition*

$$\lim_{t \to \infty} k_t = k^*,$$

where k^* *is the steady state value of capital, that is,* $f'(k^*) = \rho + \delta.$

- *Remark.* That the transversality condition reduces to the requirement that capital converges to the steady state will be argued later, with the help of the phase diagram.

5 Continuous Time and Phase Diagram

- We now want to consider the continuous-time version of the Ramsey model. This will facilitate a very convenient graphical representation of the dynamics of the economy.

- However, I will not set up and solve the continuous-time model itself. Instead, I will use a heuristic, to tell you how you can obtain the continuous-time dynamics from the discrete-time system we have derived so far.

- Consider the Euler condition with CEIS preferences. Taking logs on both side, we get

$$\ln c_{t+1} - \ln c_t = \theta \left[\ln \beta - \ln(1 + f'(k_{t+1}) - \delta).\right]$$

Next, note that $\ln \beta = -\ln(1+\rho) \approx -\rho$ and $\ln[1-\delta+f'(k)] \approx f'(k)-\delta$. We thus obtain and approximation to the Euler condition as

$$\ln c_{t+1} - \ln c_t \approx \theta[f'(k_{t+1}) - \delta - \rho].$$

This approximation turns out to be exact when time is continuous. Indeed, the continuous-time version of the Euler condition is given by

$$\frac{d \ln c}{dt} = \theta[f'(k_t) - \delta - \rho]$$

Equivalently,

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho]$$

which simply says that consumption growth is proportional to the difference between the MPK, net of depreciation, and the subjective discount rate. This condition is now in continuous time, but the economics are, of course, the same as before.

- Next, consider the resource constraint. This can be rewritten as

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t,$$

and its continuous-time version is

$$\dot{k}_t = f(k_t) - \delta k_t - c_t.$$

This simply says that the change in capital is given by net aggregate saving, which is GDP minus depreciation minus consumption.

- We thus reach the following characterization of the optimal (and equilibrium) dynamics:

Proposition 4 *Consider the continuous-time version of the model. The equilibrium path* $\{c_t, k_t\}_{t\in[0,\infty)}$ *is the unique solution to*

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho]$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t,$$

for all t*, with* $k_0 > 0$ *given and* $\lim_{t\to\infty} k_t = k^*$*, where* k^* *is the steady-state capital.*

- We can now use the phase diagram to describe the dynamics of the economy. See Figure 3.1.

- The $\dot{k} = 0$ locus is given by (c, k) such that

$$\dot{k} = f(k) - \delta k - c = 0 \quad \Leftrightarrow \quad c = f(k) - \delta k$$

On the other hand, the $\dot{c} = 0$ locus is given by (c, k) such that

$$\dot{c} = c\theta[f'(k) - \delta - \rho] = 0 \quad \Leftrightarrow \quad k = k^* \text{ or } \ c = 0$$

- The steady state is simply the intersection of the two loci:

$$\dot{c} = \dot{k} = 0 \ \Leftrightarrow \{(c, k) = (c^*, k^*) \text{ or } \ (c, k) = (0, 0)\}$$

where $k^* \equiv (f')^{-1}(\rho + \delta)$ and $c^* \equiv f(k^*) - \delta k^*$.

- We henceforth ignore the $(c, k) = (0, 0)$ steady state and the $c = 0$ part of the $\dot{c} = 0$ locus.

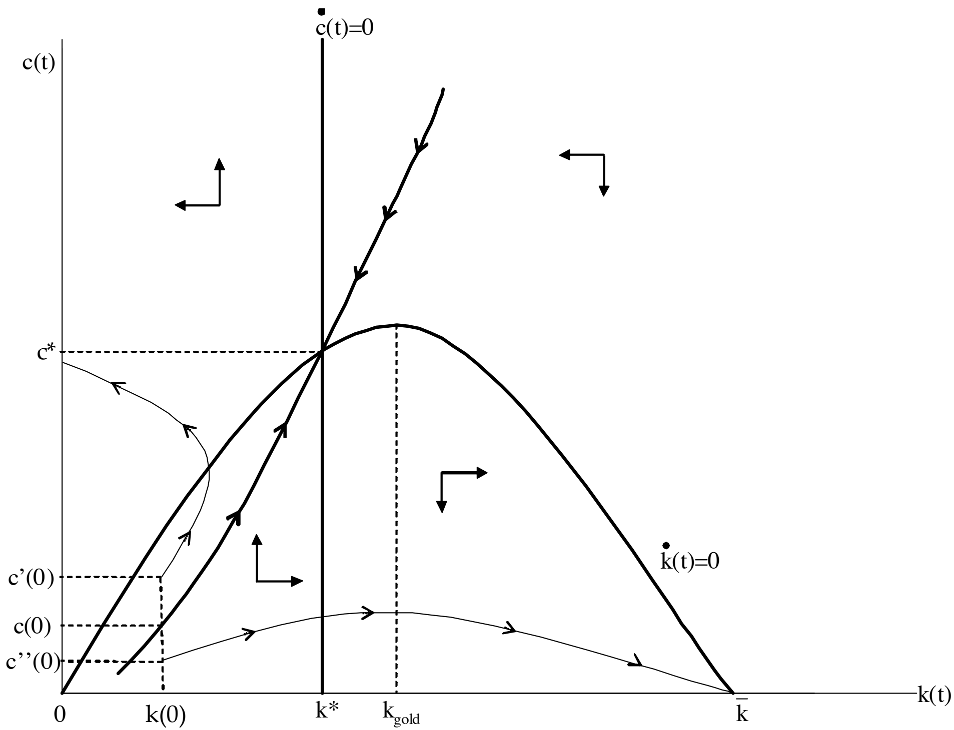

Figure 3.1 (borrowed from Acemoglu 2008)

- The two loci partition the (c, k) space in four regions. We now examine what is the direction of change in c and k in each of these four regions.

- Consider first the direction of $\dot{c}$. If $0 < k < k^*$ [resp., $k > k^*$], then and only then $\dot{c} > 0$ [resp., $\dot{c} < 0$]. That is, c increases [resp., decreases] with time whenever (c, k) lies the left [resp., right] of the $\dot{c} = 0$ locus. The direction of $\dot{c}$ is represented by the vertical arrows in Figure 3.1.

- Consider next the direction of $\dot{k}$. If $c < f(k) - \delta k$ [resp., $c > f(k) - \delta k$], then and only then $\dot{k} > 0$ [resp., $\dot{k} < 0$]. That is, k increases [resp., decreases] with time whenever (c, k) lies below [resp., above] the $\dot{k} = 0$ locus. The direction of $\dot{k}$ is represented by the horizontal arrows in Figure 3.1.

- We can now draw the time path of $\{k_t, c_t\}$ starting from any arbitrary (k_0, c_0), as in Figure 3.1. Note that there are only two such paths that go through the steady state. The one with positive slope represents the stable manifold or saddle path. The other corresponds to the unstable manifold. The equilibrium path of the economy for any initial k_0 is given by the stable manifold. That is, for any given k_0, the equilibrium c_0 is the one that puts the economy on the saddle path.

- To understand why the saddle path is the optimal path when the horizon is infinite, note the following:

  - Any c_0 that puts the economy *above* the saddle path leads to zero capital and zero consumption in finite time, thus violating the Euler condition at that time. Of course, if the horizon was finite, such a path would have been the equilibrium path. But with infinite horizon it is better to consume less and invest more in period 0, so as to never be forced to consume zero at finite time.
  - On the other hand, any c_0 that puts the economy *below* the saddle path leads to so much capital accumulation in the limit that the transversality condition is violated. Actually, in finite time the economy has crossed the golden-rule and will henceforth become dynamically inefficient. Once the economy reaches k_{gold}, where $f'(k_{gold}) - \delta = 0$, continuing on the path is dominated by an alternative feasible path, namely that of stopping investing in new capital and instead consuming $c = f(k_{gold}) - \delta k_{gold}$ thereafter. In other words, the economy is wasting too much resources in investment and it would better increase consumption.

- Let the function $c(k)$ represent the saddle path. In terms of dynamic programming, $c(k)$ is simply the optimal policy rule for consumption given capital k. Equivalently, the optimal policy rule for capital accumulation is given by

$$\dot{k} = f(k) - \delta k - c(k),$$

with the discrete-time analogue being

$$k_{t+1} = G(k_t) \equiv f(k_t) + (1-\delta)k_t - c(k_t).$$

- Finally, note that, no matter the form of $U(c)$, you can write the dynamics in terms of k and λ:

$$\begin{aligned} \frac{\dot{\lambda}_t}{\lambda_t} &= f'(k_t) - \delta - \rho \\ \dot{k}_t &= f(k_t) - \delta k_t - c(\lambda_t), \end{aligned}$$

where $c(\lambda)$ solves $U_c(c) = \lambda$, that is, $c(\lambda) \equiv U_c^{-1}(\lambda)$. Note that $U_{cc} < 0$ implies $c'(\lambda) < 0$. As an exercise, draw the phase diagram and analyze the dynamics in terms of k and λ.

6 Comparative Statics and Dynamic Responses

We will now consider how the economy responds to various changes in the environment.

6.1 Changes in the discount rate (willingness to save)

- suppose that the discount factor β falls permanently (or equivalently the discount rate ρ increases). think of this as a reduction in the willingness to save.

- the resource constraint of the economy is not affected by this change. as a result, the $\dot{k} = 0$ locus does not change.

- however, the Euler condition is affected: for any given MPK, the optimal consumption growth is lower the higher the discount rate. it follows that the $\dot{c} = 0$ shifts to the left.

- clearly, the new steady state is characterized by both a lower capital stock and a lower consumption.

- but what happens on impact? that is, starting from the old steady state, where ρ was low, how does the economy responds over time to a permanent increase in ρ?
- think about. the answer is that on impact consumption jumps up: people start eating their savings. as a consequence, the economy starts de-accumulating capital, asymptotically converging toward the new steady state.
- see Romer's textbook for a more detailed treatment of this exercise.

6.2 Additive Endowment

- Suppose that the representative household receives an endowment $e > 0$ from God, so that its budget becomes

$$c_t + k_{t+1} = w_t + r_t k_t + (1 - \delta)k_t + e$$

Adding up the budget across households gives the new resource constraint of the economy

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t + e$$

On the other hand, optimal consumption growth is given again by

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+1}) - \delta]\}^{\theta}$$

- Turning to continuous time, we conclude that the phase diagram becomes

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = f(k_t) - \delta k_t - c_t + e.$$

- In the steady state, k^* is independent of e and c^* moves one to one with e.

- Consider a permanent increase in e by Δe. This leads to a parallel shift in the $\dot{k} = 0$ locus, but no change in the $\dot{c} = 0$ locus. If the economy was initially at the steady state, then k stays constant and c simply jumps by exactly e. On the other hand, if the economy was below the steady state, c will initially increase but by less that e, so that both the level and the rate of consumption growth will increase along the transition. See Figure 3.2.

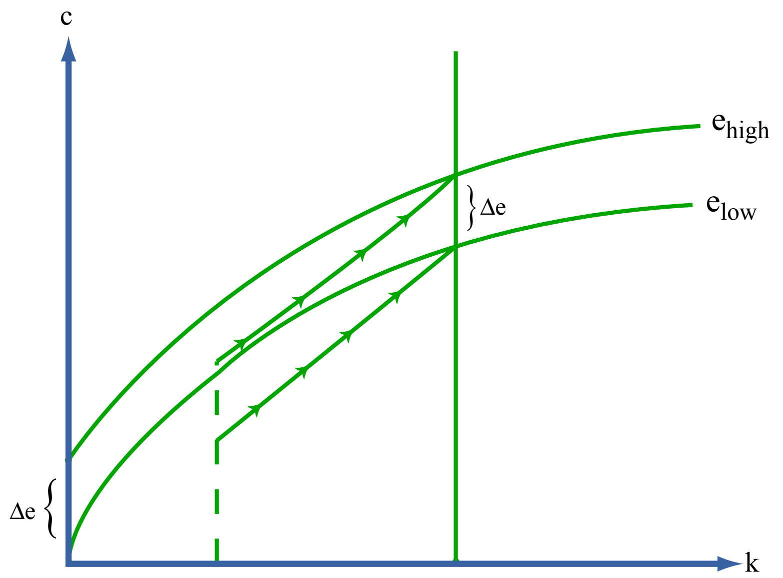

Figure 3.2

6.3 Taxation and Redistribution

- Suppose that the government taxes labor and capital income at a flat tax rate $\tau \in (0,1)$. The government then redistributes the proceeds from this tax uniformly across households. Let T_t be the transfer made in period t.

- The household budget is

$$c_t + k_{t+1} = (1-\tau)(w_t + r_t k_t) + (1-\delta)k_t + T_t,$$

implying

$$\frac{U_c(c_t)}{U_c(c_{t+1})} = \beta[1 + (1-\tau)r_{t+1} - \delta].$$

That is, the tax rate decreases the private return to investment. Combining with $r_t = f'(k_t)$ we infer

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + (1-\tau)f'(k_{t+1}) - \delta]\}^{\theta}.$$

- Adding up the budgets across household gives

$$c_t + k_{t+1} = (1-\tau)f(k_{t+1}) + (1-\delta)k_t + T_t$$

The government budget on the other hand is

$$T_t = \tau(w_t + r_t k_t) = \tau f(k_t)$$

Combining we get the resource constraint of the economy:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$$

Observe that, of course, the tax scheme does not appear in the resource constraint of the economy, for it is only redistributive and does not absorb resources.

- We conclude that the phase diagram becomes

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t.$$

- In the steady state, k^* and c^* are decreasing functions of τ.

A. Unanticipated Permanent Tax Cut

- Consider an unanticipated permanent tax cut that is enacted immediately. The $\dot{k} = 0$ locus does not change, but the $\dot{c} = 0$ locus shifts right. The saddle path thus shifts right. See Figure 3.3.

- A permanent tax cut leads to an immediate negative jump in consumption and an immediate positive jump in investment. Capital slowly increases and converges to a higher k^*. Consumption initially is lower, but increases over time, so soon it recovers and eventually converges to a higher c^*.

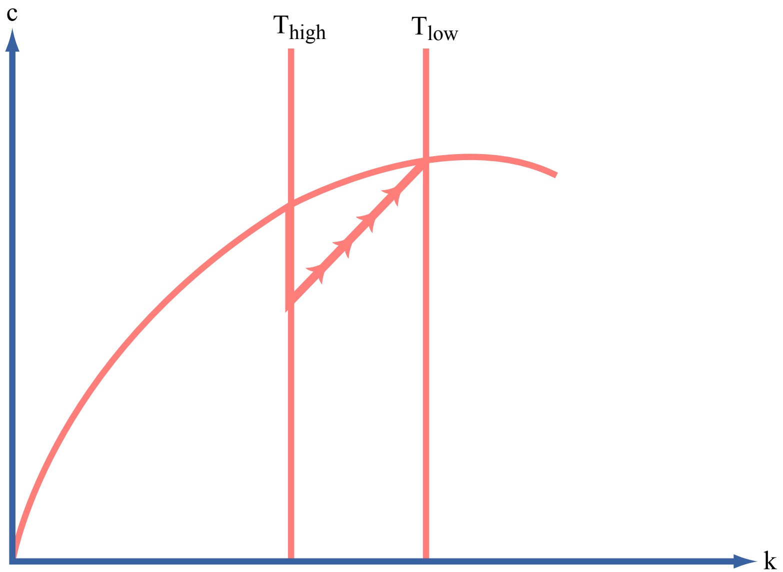

Figure 3.3

B. Anticipated Permanent Tax Cut

- Consider a permanent tax cut that is (credibly) announced at date 0 to be enacted at some date $\widehat{t} > 0$. The difference from the previous exercise is that $\dot{c} = 0$ locus now does not change immediately. It remains the same for $t < \widehat{t}$ and shifts right only for $t > \widehat{t}$. Therefore, the dynamics of c and k will be dictated by the "old" phase diagram (the one corresponding to high τ) for $t < \widehat{t}$ and by the "new" phase diagram (the one corresponding to low τ) for $t > \widehat{t}$,

- At $t = \widehat{t}$ and on, the economy must follow the saddle path corresponding to the new low τ, which will eventually take the economy to the new steady state. For $t < \widehat{t}$, the economy must follow a path dictated by the old dynamics, but at $t = \widehat{t}$ the economy must exactly reach the new saddle path. If that were not the case, the consumption path would have to jump at date $\widehat{t}$, which would violate the Euler condition (and thus be suboptimal). Therefore, the equilibrium c_0 is such that, if the economy follows a path dictated by the old dynamics, it will reach the new saddle path exactly at $t = \widehat{t}$. See Figure 3.4.

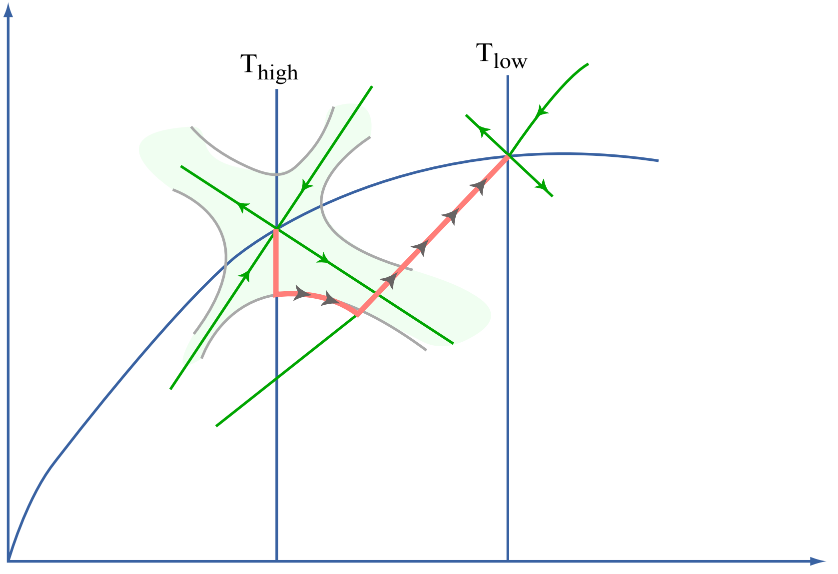

Figure 3.4

- Following the announcement, consumption jumps down and continues to fall as long as the tax cut is not initiated. The economy is building up capital in anticipation of the tax cut. As soon as the tax cut is enacted, capital continues to increase, but consumption also starts to increase. The economy then slowly converges to the new higher steady state.

6.4 Productivity Shocks: A prelude to RBC

- We now consider the effect of a shock in total factor productivity (TFP). The reaction of the economy in our deterministic framework is similar to the dynamic responses we get in a stochastic Real Business Cycle (RBC) model. Note, however, that here we consider the case that labor supply is exogenously fixed. The reaction of the economy will be different with endogenous labor supply, whether we are in the deterministic or the stochastic case.

- Let output be given by

$$y_t = A_t f(k_t)$$

where A_t denotes TFP. Note that

$$\begin{aligned} r_t &= A_t f'(k_t) \\ w_t &= A_t[f(k_t) - f'(k_t)k_t] \end{aligned}$$

so that both the return to capital and the wage rate are proportional to TFP.

- We can then write the dynamics as

$$\frac{\dot{c}_t}{c_t} = \theta[A_t f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = A_t f(k_t) - \delta k_t - c_t.$$

Note that TFP A_t affects both the production possibilities frontier of the economy (the resource constrain) and the incentives to accumulate capital (the Euler condition).

- In the steady state, both k^* and c^* are increasing in A.

A. Unanticipated Permanent Productivity Shock

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, permanently.

- c_0 may either increase or fall, depending on whether wealth or substitution effect dominates. Along the transition, both c and k are increasing towards the new higher steady state. See Figure 3.5 for the dynamics.

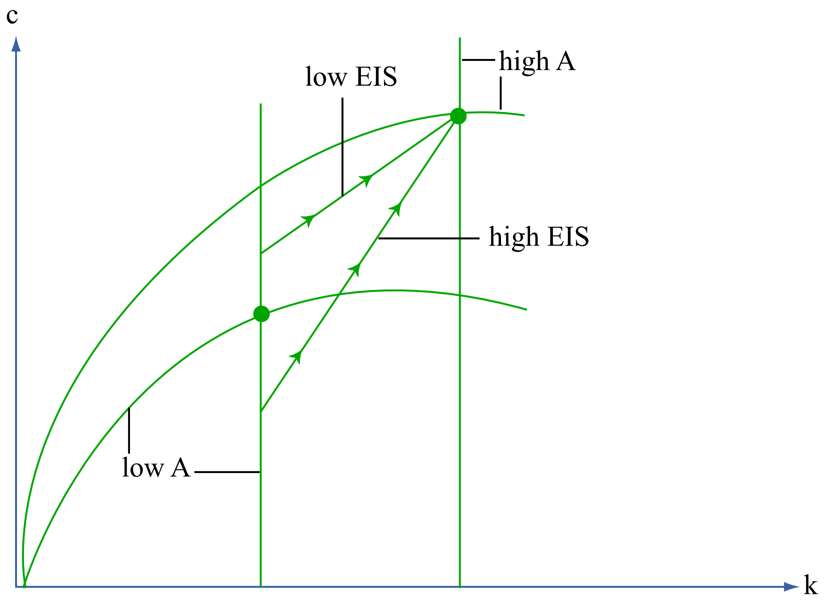

Figure 3.5 (A)

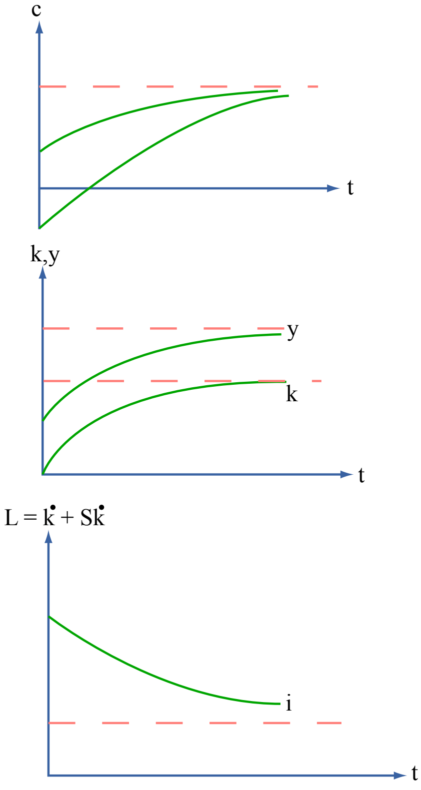

Figure 3.5 (B)

B. Unanticipated Transitory Productivity Shock [ADVANCED, you can skip]

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, but only for $t \in [0, \widehat{t}]$ for some finite $\widehat{t}$.

- Again, c_0 may either increase or fall, depending on whether wealth or substitution effects dominates. I consider the case that c_0 increases. A typical transition is depicted in Figure 3.6.

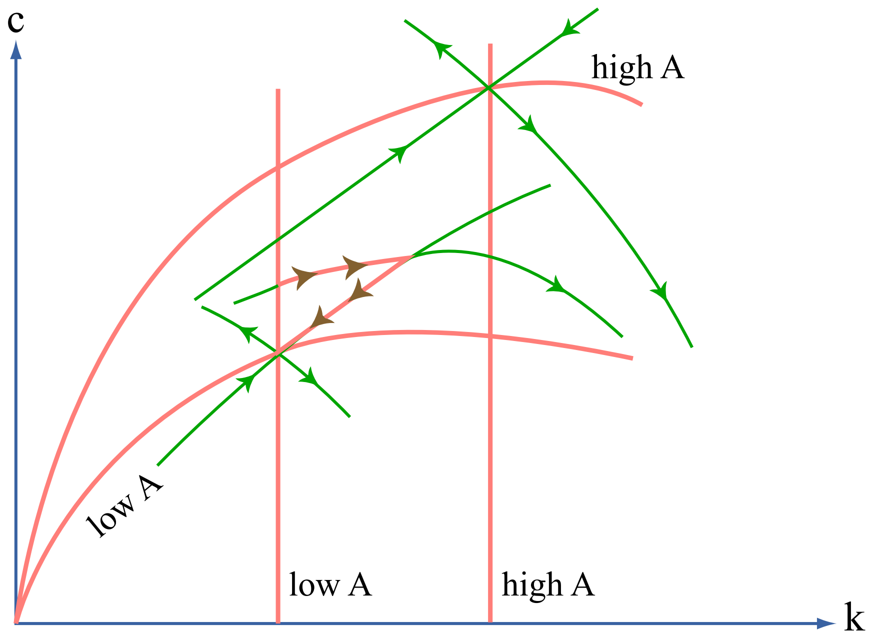

Figure 3.6 (A)

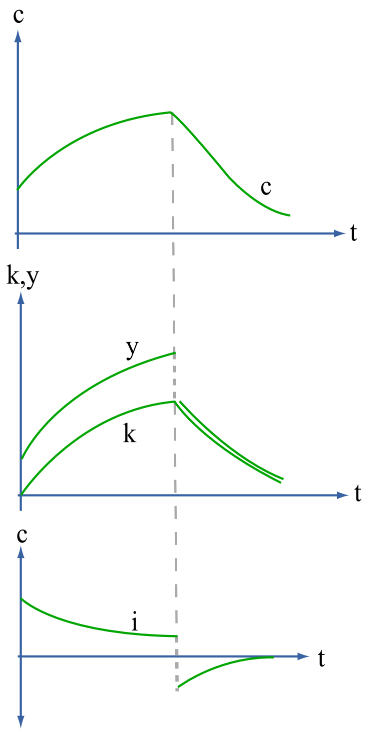

Figure 3.6 (B)

6.5 Government Spending

- We now introduce a government that collects taxes in order to finance some exogenous level of government spending.

A. Lump Sum Taxation

- Suppose the government finances its expenditure with lump-sum taxes. The household budget is

$$c_t^j + k_{t+1}^j = w_t + r_t k_t^j + (1-\delta)k_t^j - T_t,$$

implying that the Euler condition remains

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + r_{t+1} - \delta] = \beta[1 + f'(k_{t+1}) - \delta]$$

That is, taxes do not affect the savings choice.

- The government budget is $T_t = g_t$, where g_t denotes government spending.

- The resource constraint of the economy becomes

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

  Note that g_t absorbs resources from the economy.

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = f(k_t) - \delta k_t - c_t - g_t$$

- In the steady state, k^* is independent of g and c^* moves one-to-one with $-g$. Along the transition, a permanent increase in g both decreases c and slows down capital accumulation.

- Clearly, the effect of government spending financed with lump-sum taxes is isomorphic to a negative endowment shock.

B. Distortionary Taxation

- Suppose the government finances its expenditure with distortionary income taxation. The household budget is

$$c_t^j + k_{t+1}^j = (1-\tau)(w_t + r_t k_t^j) + (1-\delta)k_t^j,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + (1-\tau)r_{t+1} - \delta] = \beta[1 + (1-\tau)f'(k_{t+1}) - \delta].$$

That is, taxes now distort the savings choice.

- The government budget is

$$g_t = \tau f(k_t)$$

and the resource constraint of the economy is

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = (1-\tau)f(k_t) - \delta k_t - c_t.$$

- In the steady state, k^* is a decreasing function of g (equivalently, τ) and c^* decreases more than one-to-one with g. Along the transition, a permanent increase in g (and τ) drastically slows down capital accumulation.

- Clearly, the effect of government spending financed with distortionary taxes is isomorphic to a negative TFP shock.

14.05 Intermediate Macroeconomics
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