14.05 Lecture Notes

Introduction and The Solow Model

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1 Preliminaries

- In the real world, we observe for each country time series of macroeconomic variables such as aggregate output (GDP), consumption, investment, employment, unemployment, etc. These are the typical data that concern the macroeconomist.

- We also observe certain patterns (correlations, stylized facts) either over time or in the cross-section. For example, here is a pattern in the time-series dimension: during booms and recessions, output, consumption, investment and unemployment all move together in the same direction. And here is a pattern in the cross-section: richer countries tend to have more capital.

- Understanding what lies beneath these patterns and deriving lessons that can guide policy is the job of the macroeconomist. But "understanding" for the formal economist does not mean just telling a "story" of the short you can find in the financial news or the blogosphere. It means to develop a coherent, self-consistent, formal explanation of all the relevant observed patterns.

- To this goal, macroeconomists develop and work with mathematical models. Any such model abstracts from the infinity of forces that may be at play in the real world, focuses on a few forces that are deemed important, and seeks to work out how these forces contribute towards generating the observed patterns.

- Any such model thus features abstract concepts that are meant to mimic certain aspects of the world. There are "households" and "firms" in our models that are meant to be proxies for real-world people and businesses. And they are making choices whose product at the aggregate level is some times series for aggregate output, employment, etc. We thus end up with a mathematical model that generates the kind of times series we also observe in the real world. And by figuring how these times series are generated in the model, we hope to also understand some of the forces behind the actual macroeconomic phenomena.

- In this lecture note, we will go over our first, basic, mathematical model of the macroeconomy: the Solow model. We are going to use this model extensively to understand economic growth over time and in the cross-section of countries. But we are also going to use it to standard understanding economic fluctuations and the economic impact of various policies. All in all, we will thus see how a very simple—in fact, ridiculously simple—mathematical model can give us a lot of insight about how the macroeconomy works.

- On the way, we will also familiarize ourselves with formal notions that we will use in subsequent richer models, including the difference (or coincidence) between market outcomes and socially optimal outcomes.

- In particular, we will start analyzing the model by pretending that there is a social planner, or "benevolent dictator", that chooses the static and intertemporal allocation of resources and dictates these allocations to the households and firms of the economy. We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner (under certain assumptions).

- Be aware of the following. To talk meaningfully of a benevolent social planner, we need to have well specified preferences for the households of the economy. This is not going to be the case in the Solow model. Nevertheless, we will establish a certain isomorphism between centralized and decentralized allocations as a prelude to a similar exercise that we will undertake in the Ramsey model, where preferences are going to be well specified. This isomorphism is going to be the analogue within the Solow model of an important principle that you should know more generally for a wide class of convex economies without externalities and other market frictions: for such economies, the two welfare theorems apply, guaranteeing the set of Pareto Optimal allocations coincides with the set of Competitive Equilibria.

2 Introduction and stylized facts about growth

- *How can countries with low level of GDP per person catch up with the high levels enjoyed by the United States or the G7?*

- Only by high growth rates sustained for long periods of time.

- *Small differences in growth rates over long periods of time can make huge differences in final outcomes.*

- US per-capita GDP grew by a factor ≈ 10 from 1870 to 2000: In 1995 prices, it was \$3300 in 1870 and \$32500 in 2000.[1] Average growth rate was $\approx 1.75\%$. If US had grown with .75% (like India, Pakistan, or the Philippines), its GDP would be only \$8700 in 1990 (i.e., $\approx 1/4$ of the actual one, similar to Mexico, less than Portugal or Greece). If US had grown with 2.75% (like Japan or Taiwan), its GDP would be \$112000 in 1990 (i.e., 3.5 times the actual one).

[1] Let y_0 be the GDP per capital at year 0, y_T the GDP per capita at year T, and x the average annual growth rate over that period. Then, $y_T = (1+x)^T y_0$. Taking logs, we compute $\ln y_T - \ln y_0 = T \ln(1+x) \approx Tx$, or equivalenty $x \approx (\ln y_T - \ln y_0)/T$.

- At a growth rate of 1%, our children will have ≈ 1.4 our income. At a growth rate of 3%, our children will have ≈ 2.5 our income. Some East Asian countries grew by 6% over 1960-1990; this is a factor of ≈ 6 within just one generation!!!

- Once we appreciate the importance of sustained growth, the question is natural: *What can do to make growth faster?* Equivalently: What are the factors that explain differences in economic growth, and how can we control these factors?

- In order to prescribe policies that will promote growth, we need to understand what are the determinants of economic growth, as well as what are the effects of economic growth on social welfare. That's exactly where Growth Theory comes into picture...

2.1 The World Distribution of Income Levels and Growth Rates

- As we mentioned before, in 2000 there were many countries that had much lower standards of living than the United States. This fact reflects the high cross-country dispersion in the level of income.

- Figure 3.1 in the Barro textbook shows the distribution of GDP per capita in 2000 across the 147 countries in the Summers and Heston dataset. The richest country was Luxembourg, with $44000 GDP per person. The United States came second, with $32500. The G7 and most of the OECD countries ranked in the top 25 positions, together with Singapore, Hong Kong, Taiwan, and Cyprus. Most African countries, on the other hand, fell in the bottom 25 of the distribution. Tanzania was the poorest country, with only $570 per person – that is, less than 2% of the income in the United States or Luxemburg!

- Figure 3.2 shows the distribution of GDP per capita in 1960 across the 113 countries for which data are available. The richest country then was Switzerland, with $15000; the United States was again second, with $13000, and the poorest country was again Tanzania, with $450.

- The cross-country dispersion of income was thus as wide in 1960 as in 2000. Nevertheless, there were some important movements during this 40-year period. Argentina, Venezuela, Uruguay, Israel, and South Africa were in the top 25 in 1960, but none made it to the top 25 in 2000. On the other hand, China, Indonesia, Nepal, Pakistan, India, and Bangladesh grew fast enough to escape the bottom 25 between 1960 and 1970. These large movements in the distribution of income reflects sustained differences in the rate of economic growth.

- Figure 3.3 shows the cross-country distribution of the growth rates between 1960 and 2000. Just as there is a great dispersion in income levels, there is a great dispersion in growth rates. The mean growth rate was 1.8% per annum; that is, the world on average was twice as rich in 2000 as in 1960. The United States did slightly better than the mean. The fastest growing country was Taiwan, with a annual rate as high as 6%, which accumulates to a factor of 10 over the 40-year period. The slowest growing country was Zambia, with an negative rate at -1.8%; Zambia's residents show their income shrinking to half between 1960 and 2000.

- Most East Asian countries (Taiwan, Singapore, South Korea, Hong Kong, Thailand, China, and Japan), together with Bostwana (an outlier as compared to other sub-Saharan African countries), Cyprus, Romania, and Mauritus, had the most stellar growth performances; they were the "growth miracles" of our times. Some OECD countries (Ireland, Portugal, Spain, Greece, Luxemburg and Norway) also made it to the top 20 of the growth-rates chart. On the other hand, 18 out of the bottom 20 were sub-Saharan African countries. Other notable "growth disasters" were Venezuela, Chad and Iraq.

2.2 Stylized Facts

The following are stylized facts that should guide us in the modeling of economic growth (Kaldor, Kuznets, Romer, Lucas, Barro, Mankiw-Romer-Weil, and others):

1. *In the short run, important fluctuations:* Output, employment, investment, and consumptio vary a lot across booms and recessions.

2. *In the long run, balanced growth:* Output per worker and capital per worker (Y/L and K/L) grow at roughly constant, and certainly not vanishing, rates. The capital-to-output ratio (K/Y) is nearly constant. The return to capital (r) is roughly constant, whereas the wage rate (w) grows at the same rates as output. And, the income shares of labor and capital (wL/Y and rK/Y) stay roughly constant.

3. Substantial *cross-country differences* in both income levels and growth rates.

4. Persistent differences versus conditional convergence.

5. *Formal education:* Highly correlated with high levels of income (obviously two-direction

causality); together with differences in saving rates can "explain" a large fraction of the cross-country differences in output; an important predictor of high growth performance.

6. *R&D and IT:* Most powerful engines of growth (but require high skills at the first place).

7. *Government policies:* Taxation, infrastructure, inflation, law enforcement, property rights and corruption are important determinants of growth performance.

8. *Democracy:* An inverted U-shaped relation; that is, autarchies are bad for growht, and democracies are good, but too much democracy can slow down growth.

9. *Openness:* International trade and financial integration promote growth (but not necessarily if it is between the North and the South).

10. *Inequality:* The Kunzets curve, namely an inverted U-shaped relation between income inequality and GDP per capita (growth rates as well).

11. *Ferility:* High fertility rates correlated with levels of income and low rates of economic growth; and the process of development follows a Malthus curve, meaning that fertility rates initially increase and then fall as the economy develops.

12. *Financial markets and risk-sharing:* Banks, credit, stock markets, social insurance.

13. *Structural transformation:* agriculture→manifacture→services.

14. *Urbanization:* family production→organized production; small vilages→big cities; extended domestic trade.

15. Other institutional and social factors: colonial history, ethnic heterogeneity, social norms.

The Solow model and its various extensions that we will review in this course seek to explain how all the above factors interrelate with the process of economic growth. Once we understand better the "mechanics" of economic growth, we will be able, not only to predict economic performance for given a set of fundamentals (*positive analysis*), but also to identify what government policies or socio-economic reforms can promote social welfare in the long run (*normative analysis*).

3 The Solow Model: Centralized Allocations

- The goal here is to write a formal model of how the macroeconomy works.

- To this goal, we shall envision a central planner that takes as given the production possibilities of the economy and dictates a certain behavior to the households of the economy. As noted earlier, we will later see how the dynamics of this centralized, planning economy coincide with the dynamics of a decentralized, market economy.

- The "inputs" (or "assumptions") of the model are going to be a certain specification of the aforementioned production possibilities and behavior.

- The "output" (or "predictions") of the model will be the endogenous macroeconomic outcomes (consumption, saving, output, growth, etc.).

- We will then be able to use this model to understand the observed macroeconomic phenomena, as well as to draw policy lessons.

3.1 The Economy and the Social Planner

- Time is discrete, $t \in \{0, 1, 2, ...\}$. You can think of the period as a year, as a generation, or as any other arbitrary length of time.

- The economy is an isolated island. Many households live in this island. There are no markets and production is centralized. There is a benevolent dictator, or social planner, who governs all economic and social affairs.

- There is one good, which is produced with two factors of production, capital and labor, and which can be either consumed in the same period, or invested as capital for the next period.

- Households are each endowed with one unit of labor, which they supply inelastically to the social planner. The social planner uses the entire labor force together with the accumulated aggregate capital stock to produce the one good of the economy.

- In each period, the social planner saves a constant fraction $s \in (0, 1)$ of contemporaneous output, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the households of the economy.

- In what follows, we let L_t denote the number of households (and the size of the labor force) in period t, K_t aggregate capital stock in the beginning of period t, Y_t aggregate output in period t, C_t aggregate consumption in period t, and I_t aggregate investment in period t. The corresponding lower-case variables represent per-capita measures: $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $i_t = I_t/L_t$, and $c_t = C_t/L_t$.

3.2 Technology and Production Possibilities

- The technology for producing the good is given by

$$Y_t = F(K_t, L_t) \tag{1}$$

where $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a (stationary) production function. We assume that F is continuous and (although not always necessary) twice differentiable.

- We say that the technology is "*neoclassical*" if F satisfies the following properties

  1. Constant returns to scale (CRS), a.k.a. homogeneity of degree 1 or linear homogeneity:[2]

  $$F(\mu K, \mu L) = \mu F(K, L), \quad \forall \mu > 0.$$

  2. Positive and diminishing marginal products:

  $$F_K(K, L) > 0, \quad F_L(K, L) > 0,$$
  $$F_{KK}(K, L) < 0, \quad F_{LL}(K, L) < 0.$$

  where $F_x \equiv \partial F/\partial x$ and $F_{xz} \equiv \partial^2 F/(\partial x \partial z)$ for $x, z \in \{K, L\}$.

  3. Inada conditions:

  $$\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \infty,$$
  $$\lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0.$$

[2] We say that a function $g : \mathbb{R}^n_+ \to \mathbb{R}$ is homogeneous of degree λ if, for every vector $x \in \mathbb{R}^n_+$ and every scalar $\mu \in \mathbb{R}_+$, $g(\mu x) = \mu^\lambda g(x)$. E.g., the function $g(x) = x_1^{a_1} x_2^{a_2}$ is homogenous of degree $\lambda = a_1 + a_2$.

- By implication of CRS, F satisfies

$$Y = F(K,L) = F_K(K,L)K + F_L(K,L)L$$

  That is, total output equals the sum of the inputs times their marginal products. Equivalently, we can think of quantities $F_K(K,L)K$ and $F_L(K,L)L$ as the contributions of capital and labor into output.

- Also by CRS, the marginal products F_K and F_L are homogeneous of degree zero.[3] It follows that the marginal products depend only on the ratio K/L :

$$F_K(K,L) = F_K\left(\frac{K}{L},1\right) \qquad F_L(K,L) = F_L\left(\frac{K}{L},1\right)$$

- Finally, it must be that $F_{KL} > 0$, meaning that capital and labor are complementary inputs.[4]

[3] This is because of the more general property that, if a function is homogenous of degree λ, then its first derivatives are homogeneous of degree $\lambda - 1$.

[4] We say that two inputs are complementary if the marginal product of the one input increases with the level of the other input.

- **Technology in intensive (or per-capita) form.** Let

$$y = \frac{Y}{L} \quad \text{and} \quad k = \frac{K}{L}.$$

denote the levels of output and capital per head (or, equivalently, per worker, or per labor). Then, by CRS, we have that

$$y = f(k) \tag{2}$$

where the function f is defined by

$$f(k) \equiv F(k, 1).$$

- **Example: Cobb-Douglas production function**

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

where $\alpha \in (0, 1)$ parameterizes output's elasticity with respect to capital and $A > 0$ parameterizes TFP (total factor productivity).

In intensive form,

$$f(k) = Ak^{\alpha}$$

so that α can also be interpreted as the strength of diminishing returns: the lower α is, the more fastly the MPK, $f'(k) = \alpha k^{\alpha-1}$, falls with k.

Finally, as we will see soon, α will also coincide with the income share of labor (that is, the ratio of wL/Y) along the competitive equilibrium. This will give us a direct empirical counterpart for this theoretical parameter.

- Let us now go back to a general specification of the technology. By the definition of f and the properties of F, it is easy to show that f satisfies that following properties:

$$\begin{aligned} f(0) &= 0, \\ f'(k) &> 0 > f''(k) \\ \lim_{k\to 0} f'(k) &= \infty, \quad \lim_{k\to\infty} f'(k) = 0 \end{aligned}$$

  The first property means that output is zero when capital is zero. The second property means that the marginal product of capital (MPK) is always positive and strictly decreasing in the capital-labor ratio k. The third property means that the MPK is arbitrarily high when k is low enough, and converges to zero as k becomes arbitrarily high.

- Also, it is easy to check that

$$F_K(K,L) = f'(k) \qquad \text{and} \qquad F_L(K,L) = f(k) - f'(k)k$$

  which gives us the MPK and the MPL in terms of the intensive-form production function.

- Check **Figure 1** for a graphical representation of a typical function f.

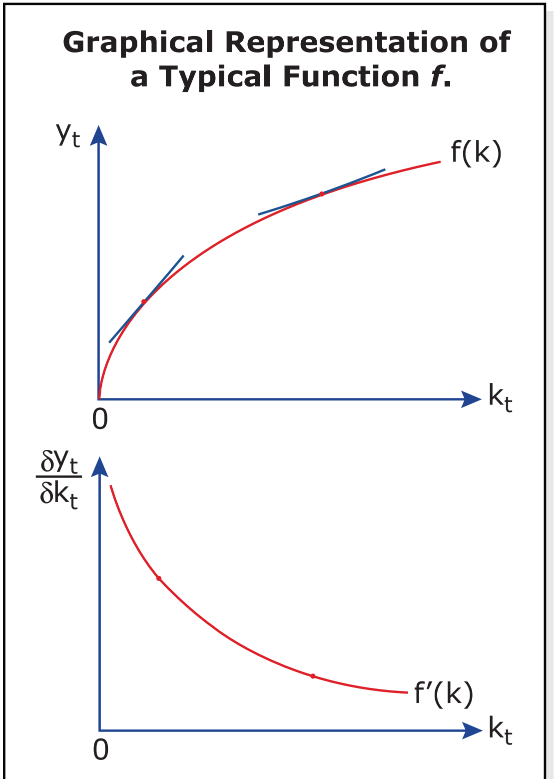


Figure 1

Image by MIT OpenCourseWare.

3.3 The Resource Constraint

- Remember that there is a single good, which can be either consumed or invested. Of course, the sum of aggregate consumption and aggregate investment can not exceed aggregate output. That is, the social planner faces the following *resource constraint*:

$$C_t + I_t \leq Y_t \tag{3}$$

  Equivalently, in per-capita terms:

$$c_t + i_t \leq y_t \tag{4}$$

- Suppose that population growth is $n \geq 0$ per period. The size of the labor force then evolves over time as follows:

$$L_t = (1+n)L_{t-1} = (1+n)^t L_0 \tag{5}$$

  We normalize $L_0 = 1$.

- Suppose that existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital, plus contemporaneous investment. That is, *the law of motion for capital* is

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{6}$$

Equivalently, in per-capita terms:

$$(1+n)k_{t+1} = (1-\delta)k_t + i_t$$

We can approximately write the above as

$$k_{t+1} \approx (1-\delta-n)k_t + i_t \tag{7}$$

The sum $\delta+n$ can thus be interpreted as the "effective" depreciation rate of per-capita capital: it represents the rate at which the per-capita level of capital will decay if aggregate saving (investment) is zero.

(Remark: The above approximation becomes arbitrarily good as the economy converges to its steady state. Also, it would have been exact if time was continuous rather than discrete.)

3.4 Consumption/Saving Behavior

- We will later derive consumption/saving choices from proper micro-foundations (well specified preferences). For now, we take a short-cut and assume that consumption is a fixed fraction $(1-s)$ of output:

$$C_t = (1-s)Y_t = (1-s)F(K_t, L_t) \tag{8}$$

  where $s \in (0,1)$. Equivalently, aggregate saving is given by a fraction s of GDP.

- Remark: in the textbook, consumption is defined as a fraction s of GDP *net of depreciation*. This makes little difference for all the economic insights we will deliver, but be aware of this minor mathematical difference.

3.5 The Aggregate Dynamics

- In most of the growth models that we will examine in this class, the key of the analysis will be to derive a dynamic system that characterizes the evolution of aggregate consumption and capital in the economy; that is, a system of difference equations in C_t and K_t (or c_t and k_t). This system is very simple in the case of the Solow model.

- Combining the law of motion for capital (6), the resource constraint (3), and the technology (1), we derive the following dynamic equation for the capital stock:

$$K_{t+1} - K_t = F(K_t, L_t) - \delta K_t - C_t \tag{9}$$

  That is, the change in the capital stock is given by aggregate output, minus capital depreciation, minus aggregate consumption.

- Combining conditions (8) and (9), we get a simple difference equation for the capital stock:

$$K_{t+1} = (1-\delta)K_t + sF(K_t, L_t) \tag{10}$$

At the same time, the law of motion for labor gives another difference equation:

$$L_{t+1} = (1+n)L_t \tag{11}$$

- Taken together, these two conditions pin down the entire dynamics of the labor force and the capital stock of the economy for any arbitrary initial levels (K_0, L_0): starting from such an initial point, we can compute the entire path $\{K_t, L_t\}$ simply by iterating on conditions (10) and (11). Once we have this path, it is straightforward to compute the paths of output, consumption, and investment simply by using the facts that $Y_t = F(K_t, L_t)$, $C_t = (1-s)Y_t$ and $I_t = sY_t$ for all t.

- We can reach a similar result in per-capita terms. Using (6), (4) and (2), we get that the capital-labor ratio satisfies the following difference equation:

$$k_{t+1} = (1 - \delta - n)k_t + sf(k_t), \tag{12}$$

Starting from an arbitrary initial k_0, the above condition alone pins down the entire path $\{k_t\}_{t=0}^{\infty}$ of the capital-labor ratio. Once we have this path, we can then get the per-capita levels of income, consumption and investment simply by the facts that

$$y_t = f(k_t), \quad c_t = (1 - s)y_t, \quad \text{and} \quad i_t = sy_t \quad \forall t \tag{13}$$

- *From this point and on, we will analyze the dynamics of the economy in per capita terms only.* Translating the results to aggregate terms is a straightforward exercise.

- We thus reach the following characterization of the planner's allocation for the Solow economy.

Proposition 1 *Given any initial point $k_0 > 0$, the dynamics of the planner's solution are given by the path $\{k_t\}_{t=0}^{\infty}$ such that*

$$k_{t+1} = G(k_t), \tag{14}$$

for all $t \geq 0$, where

$$G(k) \equiv sf(k) + (1 - \delta - n)k.$$

Equivalently, the growth rate of capital is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{15}$$

where

$$\gamma(k) \equiv s\phi(k) - (\delta + n), \quad \phi(k) \equiv f(k)/k.$$

- This result is powerful because it permits us to understand the entire macroeconomic dynamics simply by studying the properties of the function G (or equivalently the function γ).

3.6 Steady State

- A *steady state* of the economy is defined as any level k^* such that, if the economy starts with $k_0 = k^*$, then $k_t = k^*$ for all $t \geq 1$. That is, a steady state is any fixed point k^* of (14).

- A trivial steady state is $k = 0$: There is no capital, no output, and no consumption. This would not be a steady state if $f(0) > 0$. We are interested for steady states at which capital, output and consumption are all positive (and finite). We can then easily show the following:

Proposition 2 *Suppose* $\delta + n \in (0,1)$ *and* $s \in (0,1)$. *A steady state with* $k^* > 0$ *exists and is unique.*

Proof. k^* is a steady state if and only if it solves

$$k^* = G(k^*)$$

Equivalently

$$sf(k^*) = (\delta + n)k^*,$$

or

$$\phi(k^*) = \frac{\delta + n}{s} \tag{16}$$

where the function ϕ gives the output-to-capital ratio in the economy (equivalently, the average product of capital):

$$\phi(k) \equiv \frac{f(k)}{k}.$$

We infer that characterizing the steady state of the economy reduces to the simple task of characterizing the solution to equation (16). To do this, in turn, we simply need to study the properties of the function ϕ.

The properties of f, which we studied earlier, imply that ϕ is continuous (and twice differentiable), decreasing, and satisfies the Inada conditions at $k=0$ and $k=\infty$:

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$

$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

where the last two properties follow from an application of L'Hospital's rule.

The continuity of ϕ and its limit properties guarantee that equation (16) has a solution if and only if $\delta + n > 0$ and $s > 0$, which we have assumed. The monotonicity of ϕ then guarantees that the solution is unique. We conclude that a steady-state level of capital exists, is unique, and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right),$$

which completes the proof. ■

3.7 Comparative statics of the steady state

- We now turn to the comparative statics of the steady state. In order to study the impact of the level of technology, we now rewrite the production function as

$$Y = AF(K, L)$$

or, in intensive form,

$$y = Af(k)$$

where A is an exogenous scalar parameterizing TFP (total factor productivity).

Proposition 3 *Consider the steady state.*

The capital-labor ratio k^* and the per-capita level of income, y^* increase with the saving rate s and the level of productivity A, and decrease with the depreciation rate δ and the rate of population growth n.

The per-capital level of consumption, c^*, increases with A, decreases with δ and n, and is non-monotonic in s.

Proof. By the same argument as in the proof of the previous proposition, the steady-state level of the capital-labor ratio is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{sA}\right).$$

Recall that ϕ is a decreasing function. It follows from the Implicit Function Theorem that k^* is a decreasing function of $(\delta + n)/(sA)$, which proves the claims about the comparative statics of k^*. The comparative statics of y^* then follow directly from the fact that $y^* = Af(k^*)$ and the monotonicity of f. Finally, consumption is given by

$$c^* = (1 - s)f(k^*).$$

It follows that c^* increases with A and decreases with $\delta + n$, but is non-monotonic in s. ■

- *Example* If the production functions is Cobb-Douglas, namely $y = Af(k) = Ak^\alpha$, then $\phi(k) \equiv f(k)/k = k^{-(1-\alpha)}$ and therefore

$$k^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}} .y^* = A\left(\frac{sA}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} .$$

Equivalently, in logs,

$$\log y^* = \frac{1}{1-\alpha}\log A + \frac{\alpha}{1-\alpha}\log s - \frac{\alpha}{1-\alpha}\log(\delta + n)$$

Note that $\frac{1}{1-\alpha} > 1$, which means that, in steady state, output (and consumption) increases more than one-to-one with TFP. This is simply because capital also increases with TFP, so that output increases both because of the direct effect of TFP and because of its indirect effect through capital accumulation.

3.8 Transitional Dynamics

- The preceding analysis has characterized the (unique) steady state of the economy. Naturally, we are interested to know whether the economy will converge to the steady state if it starts away from it. Another way to ask the same question is whether the economy will eventually return to the steady state if an exogenous shock perturbs the economy away from the steady state. The following propositions uses the properties of the functions G and γ (defined in Proposition 1) to establish that, in the Solow model, convergence to the steady is always ensured and is indeed monotonic.

Proposition 4 *Given any initial* $k_0 \in (0, \infty)$*, the economy converges asymptotically to the steady state:*

$$\lim_{t \to \infty} k_t = k^*$$

Moreover, the transition is monotonic:

$$k_0 < k^* \;\Rightarrow\; k_0 < k_1 < k_2 < k_3 < ... < k^*$$

and

$$k_0 > k^* \;\Rightarrow\; k_0 > k_1 > k_2 > k_3 > ... > k^*$$

Finally, the growth rate γ_t *is positive and decreases over time towards zero if* $k_0 < k^*$*, while it is negative and increases over time towards zero if* $k_0 > k^*$*.*

Proof. From the properties of f, $G'(k) = sf'(k) + (1 - \delta - n) > 0$ and $G''(k) = sf''(k) < 0$. That is, G is strictly increasing and strictly concave. Moreover, $G(0) = 0$, $G'(0) = \infty$, $G(\infty) = \infty$, $G'(\infty) = (1 - \delta - n) < 1$. By definition of k^*, $G(k) = k$ iff $k = k^*$. It follows that $G(k) > k$ for all $k < k^*$ and $G(k) < k$ for all $k > k^*$. It follows that $k_t < k_{t+1} < k^*$ whenever $k_t \in (0, k^*)$ and therefore the sequence $\{k_t\}_{t=0}^{\infty}$ is strictly increasing if $k_0 < k^*$. By monotonicity, k_t converges asymptotically to some $\hat{k} \leq k^*$. By continuity of G, $\hat{k}$ must satisfy $\hat{k} = G(\hat{k})$, that is $\hat{k}$ must be a fixed point of G. But we already proved that G has a unique fixed point, which proves that $\hat{k} = k^*$. A symmetric argument proves that, when $k_0 > k^*$, $\{k_t\}_{t=0}^{\infty}$ is stricttly decreasing and again converges asymptotically to k^*. Next, consider the growth rate of the capital stock. This is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = s\phi(k_t) - (\delta + n) \equiv \gamma(k_t).$$

Note that $\gamma(k) = 0$ iff $k = k^*$, $\gamma(k) > 0$ iff $k < k^*$, and $\gamma(k) < 0$ iff $k > k^*$. Moreover, by diminishing returns, $\gamma'(k) = s\phi'(k) < 0$. It follows that $\gamma(k_t) < \gamma(k_{t+1}) < \gamma(k^*) = 0$ whenever $k_t \in (0, k^*)$ and $\gamma(k_t) > \gamma(k_{t+1}) > \gamma(k^*) = 0$ whenever $k_t \in (k^*, \infty)$. This proves that γ_t is positive and decreases towards zero if $k_0 < k^*$ and it is negative and increases towards zero if $k_0 > k^*$. ■

- **Figure 2** depicts $G(k)$, the relation between k_t and k_{t+1}. The intersection of the graph of G with the 45° line gives the steady-state capital stock k^*. The arrows represent the path $\{k_t\}_{t=}^{\infty}$ for a particular initial k_0.

- **Figure 3** depicts $\gamma(k)$, the relation between k_t and γ_t. The intersection of the graph of γ with the 45° line gives the steady-state capital stock k^*.

- The negative slope of the curve in Figure 3 (equivalently, the monotonic dynamics of the growth rate stated in the previous proposition) captures the concept of conditional convergence: if two countries have different levels of economic development (namely different k_0 and y_0) but otherwise share the same fundamental characteristics (namely share the same technologies, saving rates, depreciation rates, and fertility rates), then the poorer country will grow faster than the richer one and will eventually (asymptotically) catch up with it.

- Discuss local versus global stability: Because $\phi'(k^*) < 0$, the system is locally stable. Because ϕ is globally decreasing, the system is globally stable and transition is monotonic.

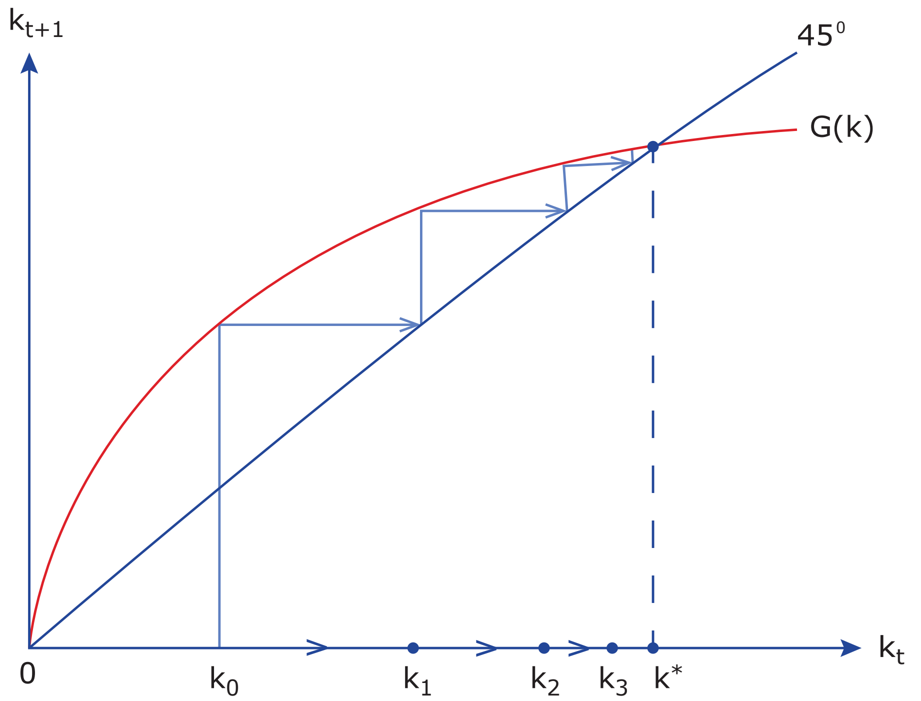


Figure 2

Image by MIT OpenCourseWare.

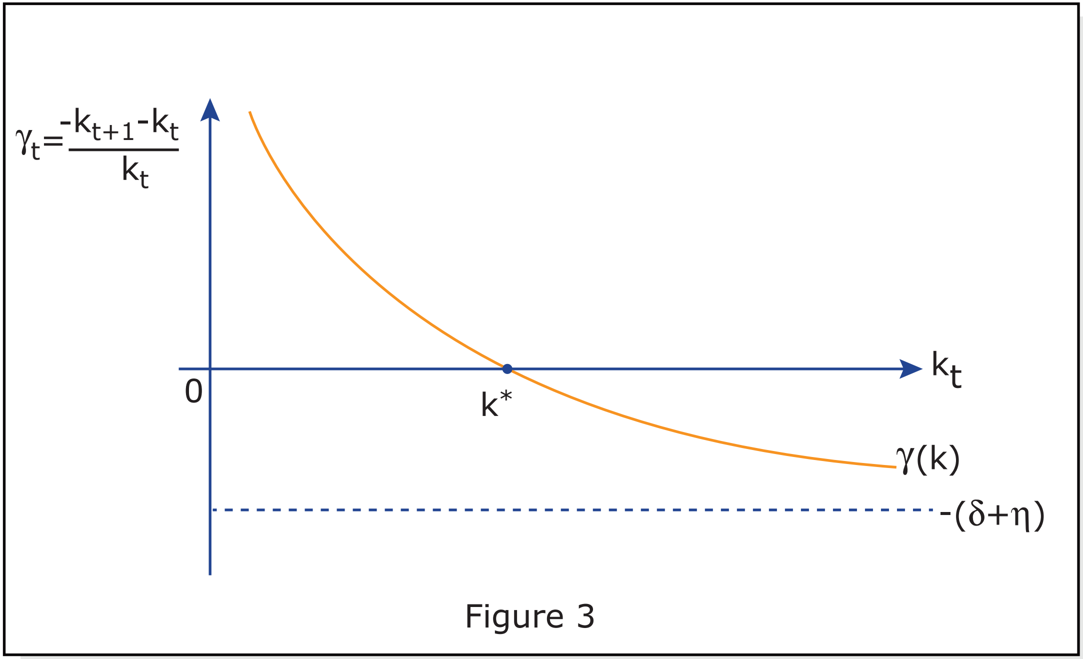


Figure 3

Image by MIT OpenCourseWare.

4 The Solow Model: Decentralized Market Allocations

- In the preceding analysis we characterized the centralized allocations dictated by a certain social planner. We now characterize the allocations chosen by the "invisible hand" of a decentralized competitive equilibrium.

4.1 Households

- Households are dynasties, living an infinite amount of time. We index households by $j \in [0, 1]$, having normalized $L_0 = 1$. The number of heads in every household grow at constant rate $n \geq 0$. Therefore, the size of the population in period t is $L_t = (1+n)^t$ and the number of persons in each household in period t is also L_t.

- We write $c_t^j, k_t^j, b_t^j, i_t^j$ for the per-head variables for household j.

- Each person in a household is endowed with one unit of labor in every period, which he supplies inelasticly in a competitive labor market for the contemporaneous wage w_t. Household j is also endowed with initial capital k_0^j. Capital in household j accumulates according to

$$(1+n)k_{t+1}^j = (1-\delta)k_t^j + i_t,$$

which we once again approximate by

$$k_{t+1}^j = (1-\delta-n)k_t^j + i_t. \tag{17}$$

- We assume that capital is owned directly by the households. But capital is productive only within firms. So we also assume that there is a competitive capital market in which firms rent the capital from the households so that they can use it as an input in their production. The capital market thus takes the form of a rental market and the per-period rental rate of capital is denoted by r_t.

- Note that this capital market is a rental market for real, physical capital (machines, buildings), not for financial contracts (funds). We are abstracting from this kind of market and also abstracting from borrowing constrain and any other form of financial frictions (frictions in how funds and resources can be channeled from one agent to another).

- The households may also hold stocks of the firms in the economy. Let π_t^j be the dividends (firm profits) that household j receive in period t. As it will become clear later on, it is without any loss of generality to assume that there is no trade of stocks. (This is because the value of firms stocks will be zero in equilibrium and thus the value of any stock transactions will be also zero.) We thus assume that household j holds a fixed fraction α^j of the aggregate index of stocks in the economy, so that $\pi_t^j = \alpha^j \Pi_t$, where Π_t are aggregate profits. Of course, $\int \alpha^j dj = 1$.

- Finally, there is also a competitive labor market, in which the households supply their labor and the firms are renting this labor to use it in their production. The wage rate (equivalently, the rental rate of labor) is denotes by w_t.

- Note that both r_t and w_t are in real terms, not nominal: they are the rental prices of capital and labor *relative* to the price of the consumption good (which has been normalized to one).

- The household uses its income to finance either consumption or investment in new capital:

$$c_t^j + i_t^j = y_t^j.$$

Total per-head income for household j in period t is simply

$$y_t^j = w_t + r_t k_t^j + \pi_t^j. \tag{18}$$

Combining, we can write the budget constraint of household j in period t as

$$c_t^j + i_t^j = w_t + r_t k_t^j + \pi_t^j \tag{19}$$

- Finally, the consumption and investment behavior of each household is assumed to follow a simple rule analogous to the one we had assumed for the social planner. They save fraction s and consume the rest:

$$c_t^j = (1-s)y_t^j \qquad \text{and} \qquad i_t^j = s y_t^i. \tag{20}$$

4.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seeks to maximize profits. The FOCs for an interior solution require

$$F_K(K_t^m, L_t^m) = r_t. \tag{21}$$

$$F_L(K_t^m, L_t^m) = w_t. \tag{22}$$

- Remember that the marginal products are homogenous of degree zero; that is, they depend only on the capital-labor ratio. In particular, F_K is a decreasing function of K_t^m/L_t^m and F_L

is an increasing function of K_t^m/L_t^m. Each of the above conditions thus pins down a unique capital-labor ratio K_t^m/L_t^m. For an interior solution to the firms' problem to exist, it must be that r_t and w_t are consistent, that is, they imply the same K_t^m/L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$r_t = f'(X_t) \tag{23}$$

$$w_t = f(X_t) - f'(X_t)X_t \tag{24}$$

where $f(k) \equiv F(k, 1)$; this follows from the properties

$$F_K(K, L) = f'(K/L) \quad \text{and} \quad F_L(K, L) = f(K/L) - f'(K/L) \cdot (K/L),$$

which we established earlier.

- If (23)-(24) are satisfied, the FOCs reduce to $K_t^m/L_t^m = X_t$, or

$$K_t^m = X_t L_t^m. \tag{25}$$

That is, the FOCs pin down the capital labor ratio for each firm (K_t^m/L_t^m), but not the size of the firm (L_t^m). Moreover, because all firms have access to the same technology, they use exactly the same capital-labor ratio.

- Besides, (23)-(24) imply

$$r_t X_t + w_t = f(X_t). \tag{26}$$

It follows that

$$r_t K_t^m + w_t L_t^m = (r_t X_t + w_t) L_t^m = f(X_t) L_t^m = F(K_t^m, L_t^m),$$

and therefore

$$\Pi_t^m = L_t^m [f(X_t) - r_t X_t - w_t] = 0. \tag{27}$$

That is, when (23)-(24) are satisfied, the maximal profits that any firm makes are exactly zero, and these profits are attained for any firm size as long as the capital-labor ratio is optimal If instead (23)-(24) were violated, then either $r_t X_t + w_t < f(X_t)$, in which case the firm could make infinite profits, or $r_t X_t + w_t > f(X_t)$, in which case operating a firm of any positive size would entail strictly negative profits.

4.3 Market Clearing

- The *capital market* clears if and only if

$$\int_0^{M_t} K_t^m dm = \int_0^1 (1+n)^t k_t^j dj$$

Equivalently,

$$\int_0^{M_t} K_t^m dm = K_t \tag{28}$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ is the aggregate capital stock in the economy.

- The *labor market*, on the other hand, clears if and only if

$$\int_0^{M_t} L_t^m dm = \int_0^1 (1+n)^t dj$$

Equivalently,

$$\int_0^{M_t} L_t^m dm = L_t \tag{29}$$

where L_t is the size of the labor force in the economy.

4.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is more meaningful when households optimize their behavior (maximize utility) rather than being automata (mechanically save a constant fraction of income). Nonetheless, it is always important to have clear in mind what is the definition of equilibrium in any model. For the decentralized version of the Solow model, we let:

Definition 5 *An equilibrium of the economy is an allocation* $\{(k_t^j, c_t^j, i_t^j)_{j\in[0,1]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=0}^{\infty}$, *a distribution of profits* $\{(\pi_t^j)_{j\in[0,1]}\}$, *and a price path* $\{r_t, w_t\}_{t=0}^{\infty}$ *such that*

(i) Given $\{r_t, w_t\}_{t=0}^{\infty}$ *and* $\{\pi_t^j\}_{t=0}^{\infty}$, *the path* $\{k_t^j, c_t^j, i_t^j\}$ *is consistent with the behavior of household* j, *for every* j.

(ii) (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t.

(iii) The capital and labor markets clear in every period

(iv) Aggregate dividends equal aggregate profits.

4.5 General Equilibrium: Existence, Uniqueness, and Characterization

- In the next, we characterize the decentralized equilibrium allocations:

Proposition 6 *For any initial positions* $(k_0^j)_{j\in[0,1]}$*, an equilibrium exists. The allocation of production across firms is indeterminate, but the equilibrium is unique as regards aggregate and household allocations. The capital-labor ratio in the economy is given by* $\{k_t\}_{t=0}^{\infty}$ *such that*

$$k_{t+1} = G(k_t) \tag{30}$$

for all $t \geq 0$ *and* $k_0 = \int k_0^j dj$ *historically given, where* $G(k) \equiv sf(k) + (1-\delta-n)k$*. Equilibrium growth is given by*

$$\gamma_t \equiv \frac{k_{t+1}-k_t}{k_t} = \gamma(k_t), \tag{31}$$

where $\gamma(k) \equiv s\phi(k) - (\delta+n)$*,* $\phi(k) \equiv f(k)/k$*. Finally, equilibrium prices are given by*

$$r_t = r(k_t) \equiv f'(k_t), \tag{32}$$

$$w_t = w(k_t) \equiv f(k_t) - f'(k_t)k_t, \tag{33}$$

where $r'(k) < 0 < w'(k)$*.*

Proof. We first characterize the equilibrium, assuming it exists.

Using $K_t^m = X_t L_t^m$ by (25), we can write the aggregate demand for capital as

$$\int_0^{M_t} K_t^m dm = X_t \int_0^{M_t} L_t^m dm$$

From the labor market clearing condition (29),

$$\int_0^{M_t} L_t^m dm = L_t.$$

Combining, we infer

$$\int_0^{M_t} K_t^m dm = X_t L_t,$$

and substituting in the capital market clearing condition (28), we conclude

$$X_t L_t = K_t,$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ denotes the aggregate capital stock.

Equivalently, letting $k_t \equiv K_t/L_t$ denote the capital-labor ratio in the economy, we have

$$X_t = k_t. \tag{34}$$

That is, all firms use the same capital-labor ratio as the aggregate of the economy.

Substituting (34) into (23) and (24) we infer that equilibrium prices are given by

$$\begin{aligned} r_t &= r(k_t) \equiv f'(k_t) = F_K(k_t, 1) \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t = F_L(k_t, 1) \end{aligned}$$

Note that $r'(k) = f''(k) = F_{KK} < 0$ and $w'(k) = -f''(k)k = F_{LK} > 0$. That is, the interest rate is a decreasing function of the capital-labor ratio and the wage rate is an increasing function of the capital-labor ratio. The first properties reflects diminishing returns, the second reflects the complementarity of capital and labor.

Adding up the budget constraints of the households, we get

$$C_t + I_t = r_t K_t + w_t L_t + \int \pi_t^j dj,$$

where $C_t \equiv \int c_t^j dj$ and $I_t \equiv \int i_t^j dj$. Aggregate dividends must equal aggregate profits, $\int \pi_t^j dj = \int \Pi_t^m dj$. By (27), profits for each firm are zero. Therefore, $\int \pi_t^j dj = 0$, implying

$$C_t + I_t = Y_t = r_t K_t + w_t L_t$$

Equivalently, in per-capita terms,

$$c_t + i_t = r_t k_t + w_t.$$

From (26) and (34), or equivalently from (32) and (33), $r_t k_t + w_t = y_t = f(k_t)$. We conclude that the household budgets imply

$$c_t + i_t = f(k_t),$$

which is simply the resource constraint of the economy.

Adding up the individual capital accumulation rules (17), we get the capital accumulation rule for the aggregate of the economy. In per-capita terms,

$$k_{t+1} = (1 - \delta - n)k_t + i_t$$

Adding up (20) across household, we similarly infer

$$i_t = sy_t = sf(k_t).$$

Combining, we conclude

$$k_{t+1} = sf(k_t) + (1 - \delta - n)k_t = G(k_t),$$

which is exactly the same as in the centralized allocation.

Finally, existence and uniqueness is now trivial. (30) maps any $k_t \in (0, \infty)$ to a unique $k_{t+1} \in (0, \infty)$. Similarly, (32) and (33) map any $k_t \in (0, \infty)$ to unique $r_t, w_t \in (0, \infty)$. Therefore, given any initial $k_0 = \int k_0^j dj$, there exist unique paths $\{k_t\}_{t=0}^{\infty}$ and $\{r_t, w_t\}_{t=0}^{\infty}$. Given $\{r_t, w_t\}_{t=0}^{\infty}$, the allocation $\{k_t^j, c_t^j, i_t^j\}$ for any household j is then uniquely determined by (17), (18), and (20). Finally, any allocation $(K_t^m, L_t^m)_{m \in [0, M_t]}$ of production across firms in period t is consistent with equilibrium as long as $K_t^m = k_t L_t^m$. ■

- An immediate implication is that the decentralized market economy and the centralized dictatorial economy are isomorphic:

Corollary 7 *The aggregate and per-capita allocations in the competitive market economy coincide with those in the dictatorial economy.*

- Given this isomorphism, we can immediately translate the steady state and the transitional dynamics of the centralized plan to the steady state and the transitional dynamics of the decentralized market allocations:

Corollary 8 *Suppose $\delta + n \in (0,1)$ and $s \in (0,1)$. A steady state $(c^*, k^*) \in (0,\infty)^2$ for the competitive economy exists and is unique, and coincides with that of the social planner. k^* and y^* increase with s and decrease with δ and n, whereas c^* is non-monotonic with s and decreases with δ and n. Finally, $y^*/k^* = (\delta + n)/s$.*

Corollary 9 *Given any initial $k_0 \in (0,\infty)$, the competitive economy converges asymptotically to the steady state. The transition is monotonic. The equilibrium growth rate is positive and decreases over time towards zero if $k_0 < k^*$; it is negative and increases over time towards zero if $k_0 > k^*$.*

- The bottom line is that the *allocations* that characterize the frictionless competitive equilibrium coincide with those that characterize the planner's solution. The only extra knowledge we got by considering the equilibrium is that we found the prices (wages, rental rates) that "support" the allocation as a decentralized market outcome, that is, that make this allocation individually optimal in the eyes of firms and households. Keep this in mind: the planner's solution is merely an allocation, a market equilibrium is always a combination of an allocation and of prices that support this allocation.

- By finding the prices that support the planner's solution as a market equilibrium, we can thus make predictions, not only about the real macroeconomic quantities (GDP, investment, consumption, etc) but also about wages, interest rates, and more generally market prices.

- Finally, remember that all this presumes that we have a competitive market economy without externalities and without any kind of friction to drive the equilibrium away from the planner's solution. If we were to allow for, say, externalities or monopoly power, the equilibrium would differ from the planner's solution—and then we could start making sense of policies that seek to correct the underlying market inefficiencies. We will consider such situations in due course.

5 The Solow Model: Introducing Shocks and Policies

- The Solow model can be used to understand business cycles (economic fluctuations).

- To do this, we must first extend the model in a way that it can accommodate stochasticity in its equilibrium outcomes. This is done by introducing exogenous random disturbances in the primitives of the model (technologies, preferences, etc). This means that we model the "deeper origins" of booms and recession as exogenous forces and then use the model to make predictions about how the endogenous macroeconomic variables respond over time to these exogenous disturbances.

- In the sequel, we do this kind of exercise to predict the response of the economy to productivity shocks (changes in the production possibilities of the economy, taste shocks (changes in the saving rate), and policy shocks (changes in government policies).

5.1 Productivity (TFP) Shock

- Let us introduce exogenous shocks to the Total Factor Productivity (TFP) of the economy. To this goal, we modify the production function as follows:

  $$Y_t = A_t F(K_t, L_t)$$

  or, in intensive form,

  $$y_t = A_t f(k_t)$$

  where A_t identifies TFP in period t.

- We thus want to consider the possibility that A_t varies over time and to examine how the economy responds to changes in A_t, according to our model. Before we do this, let us first show that variation in A_t is not merely a theoretical possibility; it is an actual fact in US data.

- To this goal, suppose further that F takes a Cobb-Douglas form: $F(K, L) = K^\alpha L^{1-\alpha}$. It follows that

  $$\log Y_t = \log A_t + \alpha \log K_t + (1-\alpha) \log L_t$$

and therefore

$$\Delta \log A_t = \Delta \log Y_t - \alpha \Delta \log K_t - (1-\alpha)\Delta \log L_t$$

where $\Delta X_t \equiv X_t - X_{t-1}$ for any variable X . Note that $\Delta \log Y_t$ is the growth rate of GDP, $\Delta \log K_t$ is the change in capital (net investment), and $\Delta \log L_t$ is the net change in employment. For all these variables, we have readily available data in the US. Furthermore, under the assumption of perfect competition, $w_t = A_t F_L(K_t, L_t)$. Given the Cobb-Douglas specification, this gives

$$w_t = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} = (1-\alpha)Y_t/L_t$$

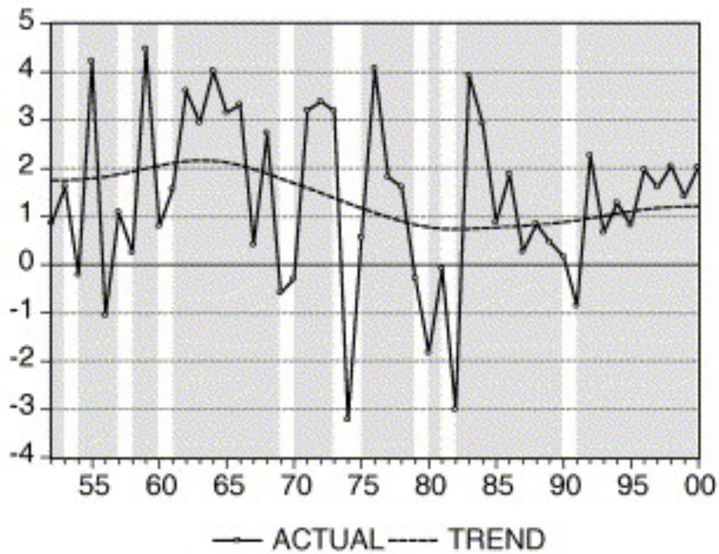
and therefore

$$1-\alpha = \frac{w_t L_t}{Y_t}$$

which means that $1-\alpha$ coincides with the income share of labor. In the US data, the income share of labor is about 70%. It follows that $\alpha \approx .3$. We conclude that the change in TFP can be computed by using the available macro data along with the following equation

$$\Delta \log A_t = \Delta \log Y_t - .3\Delta \log K_t - .7\Delta \log L_t$$

If you do this, you get a times series for $\Delta \log A_t$, the growth rate in TFP, that looks as in the following figure.



- There are two notable features in this figure. The first is that on average ΔA_t is positive. This means that on average there is long-run technological progress: out of the same inputs, we get more and more output as time passes. The second is that ΔA_t fluctuates a lot around its trend and tends to be lower during recessions (periods highlighted by the white areas in the figure) as opposed to normal times (grey areas in the figure).

- This second systematic feature, that TFP tends to fall during recessions, motivates the exercise we do here. We take for given that A_t fluctuates over time and study the model's predictions regarding how all other macroeconomic variables (output, investment, consumption) respond to such fluctuations in A_t. We are thus interested to see if the model makes reasonable and empirically plausible predictions about the cyclical behavior of these variables.

- In particular, we know that, in the data, TFP, output, consumption and investment all fall during recessions. In the model, we will *only* assume that TFP falls during recession. We then ask whether the model *predicts* that output, consumption, and investment must fall in response to a fall in TFP.

- Thus consider a negative shock in TFP. This shock could be either temporary or permanent. Also, keep in mind this shock can be interpreted literally as a change in the know-how of firms and the talents of people; but it could also be proxy from changes in the efficiency of the financial system and more generally in the efficiency of how resources are used in the economy.

- Recall that the dynamics of capital are given by

$$k_{t+1} = G(k_t; A_t) = sA_t f(k_t) + (1 - \delta - n)k_t$$

  As a result of the drop in A_t, the G function shifts down. If the drop in A_t is permanent, the shift in G is also permanent; if the drop in A_t is transitory, the shift in G is also transitory. The same logic applies if we look at the γ function, which gives the growth rate of the economy. See Figure 4.

- Suppose now that the economy was resting at its steady state before the drop in A_t. At the moment A_t falls, output falls by exactly the same amount, because at the moment resources are fixed and simply TFP has fallen. But the drop in output leads to a drop in investment, which in turn leads to lower capital stock in the future. It follows that after the initial shock there are further and further reductions in output, due to the endogenous reduction in the capital stock. In other words, the endogenous response of capital *amplifies* the effects of the negative TFP shock on output. Furthermore, if at some point the TFP shock disappears and A_t returns to its initial value, output (and by implication consumption and investment) do not return immediately to their initial values. Rather, because the capital stock has been decreased, it takes time for output to transit back to its original, pre-recession value. In this sense, the endogenous response of capital, not only amplifies the recessionary effects of the exogenous TFP shock, but it also adds *persistence*: the effects of the shock are felt in the economy long time after the shock has itself gone away. Equivalently, recoveries take time.

- The aforementioned dynamics are illustrated in Figure 5. The solid lines represent the response of the economy to a transitory negative TFP shock, which only lasts between t_1 and t_2 in the figure. The dashed lines show what the response would have been in the case the shock were permanent, starting at t_1 and lasting for ever.

either temporarily or permanently. What are the effects on the steady state and the transitional dynamics, in either case?

- See Figures 4 and 5 for a graphical representation of the impact of a (temporary) negative productivity shock.

- *Taste shocks:* Consider a temporary fall in the saving rate. The $\gamma(k)$ function shifts down for a while, and then return to its initial position. What the transitional dynamics?

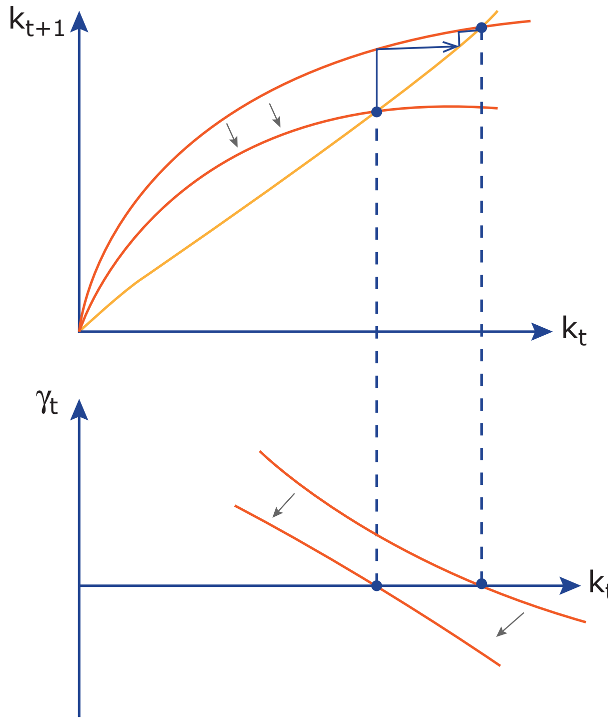


Figure 4

Image by MIT OpenCourseWare.

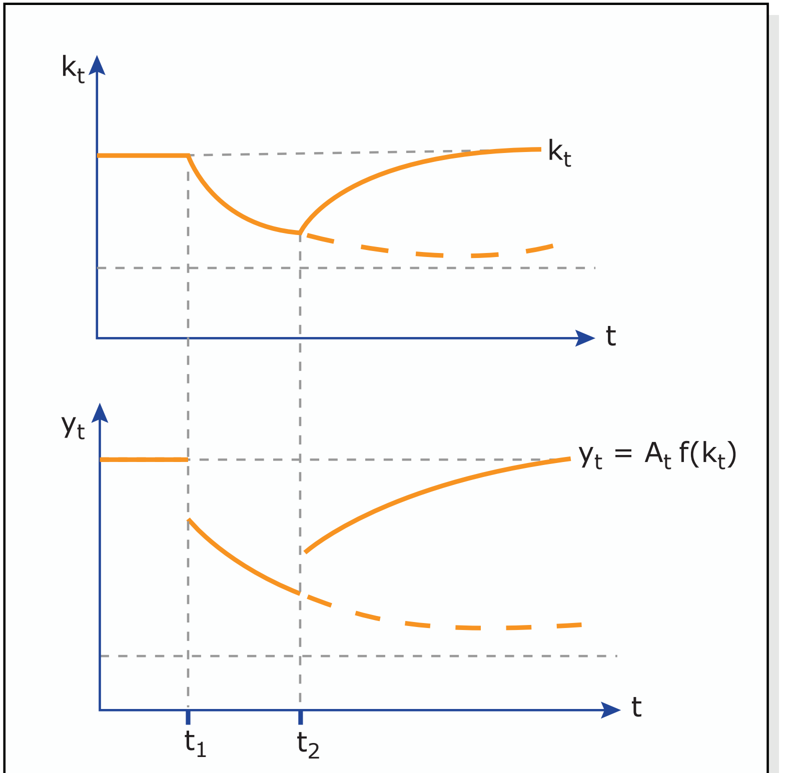


Figure 5

Image by MIT OpenCourseWare.

5.2 Unproductive Government Spending

- Let us now introduce a *government* in the competitive market economy. The government spends resources without contributing to production or capital accumulation.

- The resource constraint of the economy now becomes

$$c_t + g_t + i_t = y_t = f(k_t),$$

where g_t denotes government consumption. It follows that the dynamics of capital are given by

$$k_{t+1} - k_t = f(k_t) - (\delta + n)k_t - c_t - g_t$$

- Government spending is financed with proportional income taxation, at rate $\tau \geq 0$. The government thus absorbs a fraction τ of aggregate output:

$$g_t = \tau y_t.$$

- Disposable income for the representative household is $(1-\tau)y_t$. We continue to assume that consumption and investment absorb fractions $1-s$ and s of disposable income:

$$c_t = (1-s)(1-\tau)y_t.$$

- Combining the above, we conclude that the dynamics of capital are now given by

$$\gamma_t = \frac{k_{t+1}-k_t}{k_t} = s(1-\tau)\phi(k_t) - (\delta + n).$$

  where $\phi(k) \equiv f(k)/k$. Given s and k_t, the growth rate γ_t decreases with τ.

- A steady state exists for any $\tau \in [0,1)$ and is given by

$$k^* = \phi^{-1}\left(\frac{\delta+n}{s(1-\tau)}\right).$$

  Given s, k^* decreases with τ.

- *Policy Shocks:* Consider a temporary shock in government consumption. What are the transitional dynamics?

5.3 Productive Government Spending

- Suppose now that production is given by

$$y_t = f(k_t, g_t) = k_t^\alpha g_t^\beta,$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. Government spending can thus be interpreted as infrastructure or other productive services. The resource constraint is

$$c_t + g_t + i_t = y_t = f(k_t, g_t).$$

- We assume again that government spending is financed with proportional income taxation at rate τ, and that private consumption and investment are fractions $1 - s$ and s of disposable household income:

$$\begin{aligned} g_t &= \tau y_t. \\ c_t &= (1-s)(1-\tau) y_t \\ i_t &= s(1-\tau) y_t \end{aligned}$$

- Substituting $g_t = \tau y_t$ into $y_t = k_t^\alpha g_t^\beta$ and solving for y_t, we infer

$$y_t = k_t^{\frac{\alpha}{1-\beta}} \tau^{\frac{\beta}{1-\beta}} \equiv k_t^a \tau^b$$

  where $a \equiv \alpha/(1-\beta)$ and $b \equiv \beta/(1-\beta)$. Note that $a > \alpha$, reflecting the complementarity between government spending and capital.

- We conclude that the growth rate is given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\tau^b k_t^{a-1} - (\delta + n).$$

  The steady state is

$$k^* = \left(\frac{s(1-\tau)\tau^b}{\delta + n} \right)^{1/(1-a)}.$$

- Consider the rate τ that maximizes either k^*, or γ_t for any given k_t. This is given by

$$\frac{d}{d\tau}[(1-\tau)\tau^b] = 0 \Leftrightarrow$$

$$b\tau^{b-1} - (1+b)\tau^b = 0 \Leftrightarrow$$

$$\tau = b/(1+b) = \beta.$$

That is, the "optimal" τ equals the elasticity of production with respect to government services. The more productive government services are, the higher their optimal provision.

6 The Solow Model: Miscellaneous

6.1 The Solow Model in Continuous Time

- Recall that the basic growth equation in the discrete-time Solow model is

$$\frac{k_{t+1} - k_t}{k_t} = \gamma(k_t) \equiv s\phi(k_t) - (\delta + n).$$

  We would expect a similar condition to hold under continuous time. We verify this below.

- The resource constraint of the economy is

$$C + I = Y = F(K, L).$$

  In per-capita terms,

$$c + i = y = f(k).$$

- Clearly, these conditions do not depend on whether time is continuous or discrete. Rather, it is the law of motions for L and K that slightly change from discrete to continuous time.

- Population growth is now given by

$$\frac{\dot{L}}{L} = n$$

and the law of motion for aggregate capital is

$$\dot{K} = I - \delta K$$

- Let $k \equiv K/L$. Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Substituting from the above, we infer

$$\dot{k} = i - (\delta + n)k.$$

Combining this with

$$i = sy = sf(k),$$

we conclude

$$\dot{k} = sf(k) - (\delta + n)k.$$

- Equivalently, the growth rate of the economy is given by

$$\frac{\dot{k}}{k} = \gamma(k) \equiv s\phi(k) - (\delta + n). \tag{35}$$

  The function $\gamma(k)$ thus gives the growth rate of the economy in the Solow model, whether time is discrete or continuous.

6.2 Mankiw-Romer-Weil: Cross-Country Differences

- The Solow model implies that steady-state capital, productivity, and income are determined primarily by technology (f and δ), the national saving rate (s), and population growth (n).

- Suppose that countries share the same technology in the long run, but differ in terms of saving behavior and fertility rates. If the Solow model is correct, observed cross-country income and productivity differences should be explained by observed cross-country differences in s and n,

- Mankiw, Romer and Weil tests this hypothesis against the data. In it's simple form, the Solow model fails to explain the large cross-country dispersion of income and productivity levels.

- Mankiw, Romer and Weil then consider an extension of the Solow model, that includes two types of capital, physical capital (k) and human capital (h). The idea is to take a broader perspective on how to map the model to reality.

- Output is given by

$$y = k^{\alpha} h^{\beta},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. The dynamics of capital accumulation are now given by

$$\begin{aligned} \dot{k} &= s_k y - (\delta + n)k \\ \dot{h} &= s_h y - (\delta + n)h \end{aligned}$$

where s_k and s_h are the investment rates in physical capital and human capital, respectively. The steady-state levels of k, h, and y then depend on both s_k and s_h, as well as δ and n.

- Proxying s_h by education attainment levels in each country, Mankiw, Romer and Weil find that the Solow model extended for human capital does a pretty good job in explaining the cross-country dispersion of output and productivity levels.

6.3 Log-linearization and the Convergence Rate

- Define $z \equiv \ln k - \ln k^*$. We can rewrite the growth equation (35) as

$$\dot{z} = \Gamma(z),$$

where

$$\Gamma(z) \equiv \gamma(k^* e^z) \equiv s\phi(k^* e^z) - (\delta + n)$$

Note that $\Gamma(z)$ is defined for all $z \in \mathbb{R}$. By definition of k^*, $\Gamma(0) = s\phi(k^*) - (\delta + n) = 0$. Similarly, $\Gamma(z) > 0$ for all $z < 0$ and $\Gamma(z) < 0$ for all $z > 0$. Finally, $\Gamma'(z) = s\phi'(k^* e^z)k^* e^z < 0$ for all $z \in \mathbb{R}$.

- We next (log)linearize $\dot{z} = \Gamma(z)$ around $z = 0$:

$$\dot{z} = \Gamma(0) + \Gamma'(0) \cdot z$$

or equivalently

$$\dot{z} = \lambda z$$

where we substituted $\Gamma(0) = 0$ and let $\lambda \equiv \Gamma'(0)$.

- Straightforward algebra gives

$$\begin{aligned}\Gamma'(z) &= s\phi'(k^* e^z)k^* e^z < 0 \\ \phi'(k) &= \frac{f'(k)k - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right]\frac{f(k)}{k^2} \\ sf(k^*) &= (\delta + n)k^*\end{aligned}$$

We infer

$$\Gamma'(0) = -(1 - \varepsilon_K)(\delta + n) < 0$$

where $\varepsilon_K \equiv F_K K/F = f'(k)k/f(k)$ is the elasticity of production with respect to capital, evaluated at the steady-state k.

- We conclude that

$$\frac{\dot{k}}{k} = \lambda \ln\left(\frac{k}{k^*}\right)$$

where

$$\lambda = -(1 - \varepsilon_K)(\delta + n) < 0$$

The quantity $-\lambda$ is called the *convergence rate.*

- Note that, around the steady state

$$\frac{\dot{y}}{y} = \varepsilon_K \cdot \frac{\dot{k}}{k}$$

and

$$\frac{y}{y^*} = \varepsilon_K \cdot \frac{k}{k^*}$$

It follows that

$$\frac{\dot{y}}{y} = \lambda \ln\left(\frac{y}{y^*}\right)$$

Thus, $-\lambda$ is the convergence rate for either capital or output.

- In the Cobb-Douglas case, $y = k^\alpha$, the convergence rate is simply

$$-\lambda = (1-\alpha)(\delta + n),$$

where α is the income share of capital. Note that as $\lambda \to 0$ as $\alpha \to 1$. That is, convergence becomes slower and slower as the income share of capital becomes closer and closer to 1. Indeed, if it were $\alpha = 1$, the economy would a balanced growth path.

- In the example with productive government spending, $y = k^{\alpha} g^{\beta} = k^{\alpha/(1-\beta)} \tau^{\beta/(1-\beta)}$, we get

$$-\lambda = \left(1 - \frac{\alpha}{1-\beta}\right)(\delta + n)$$

The convergence rate thus decreases with β, the productivity of government services. And $\lambda \to 0$ as $\beta \to 1 - \alpha$.

- *Calibration:* If $\alpha = 35\%$, $n = 3\%$ (= 1% population growth+2% exogenous technological process), and $\delta = 5\%$, then $-\lambda = 6\%$. This contradicts the data. But if $\alpha = 70\%$, then $-\lambda = 2.4\%$, which matches the date.

6.4 Barro: Conditional Convergence

- Recall the log-linearization of the dynamics around the steady state:

$$\frac{\dot{y}}{y} = \lambda \ln \frac{y}{y^*}.$$

A similar relation will hold true in the neoclassical growth model a la Ramsey-Cass-Koopmans. $\lambda < 0$ reflects local diminishing returns. Such local diminishing returns occur even in endogenous-growth models. The above thus extends well beyond the simple Solow model.

- Rewrite the above as

$$\Delta \ln y = \lambda \ln y - \lambda \ln y^*$$

Next, let us proxy the steady state output by a set of country-specific controls X, which include s, δ, n, τ etc. That is, let

$$-\lambda \ln y^* \approx \beta' X.$$

We conclude

$$\Delta \ln y = \lambda \ln y + \beta' X + error$$

- The above represents a typical "Barro" conditional-convergence regression: We use cross-country data to estimate λ (the convergence rate), together with β (the effects of the saving rate, education, population growth, policies, etc.) The estimated convergence rate is about 2% per year.

- Discuss the effects of the other variables (X).

6.5 The Golden Rule and Dynamic Inefficiency

- *The Golden Rule:* Consumption at the steady state is given by

$$\begin{aligned} c^* &= (1-s)f(k^*) = \\ &= f(k^*) - (\delta+n)k^* \end{aligned}$$

Suppose the social planner chooses s so as to maximize c^*. Since k^* is a monotonic function of s, this is equivalent to choosing k^* so as to maximize c^*. Note that

$$c^* = f(k^*) - (\delta+n)k^*$$

is strictly concave in k^*. The FOC is thus both necessary and sufficient. c^* is thus maximized if and only if $k^* = k_{gold}$, where k_{gold} solve

$$f'(k_{gold}) - \delta = n.$$

Equivalently, $s = s_{gold}$, where s_{gold} solves

$$s_{gold} \cdot \phi\left(k_{gold}\right) = (\delta+n)$$

The above is called the "*golden rule*" for savings, after Phelps.

- *Dynamic Inefficiency:* If $s > s_{gold}$ (equivalently, $k^* > k_{gold}$), the economy is dynamically inefficient: If the saving raised is lowered to $s = s_{gold}$ for all t, then consumption in all periods will be higher!

- On the other hand, if $s < s_{gold}$ (equivalently, $k^* > k_{gold}$), then raising s towards s_{gold} will increase consumption in the long run, but at the cost of lower consumption in the short run. Whether such a trade-off between short-run and long-run consumption is desirable will depend on how the social planner weight the short run versus the long run.

- *The Modified Golden Rule:* In the Ramsey model, this trade-off will be resolved when k^* satisfies the

$$f'(k^*) - \delta = n + \rho,$$

where $\rho > 0$ measures impatience (ρ will be called "the discount rate"). The above is called the "*modified golden rule.*" Naturally, the distance between the Ramsey-optimal k^* and the golden-rule k_{gold} increase with ρ.

- *Abel et. al.:* Note that the golden rule can be restated as

$$r - \delta = \frac{\dot{Y}}{Y}.$$

Dynamic inefficiency occurs when $r - \delta < \dot{Y}/Y$, dynamic efficiency is ensured if $r - \delta > \dot{Y}/Y$. Abel et al. use this relation to argue that, in reality, there is no evidence of dynamic inefficiency.

- *Bubbles:* If the economy is dynamically inefficient, there is room for bubbles.

6.6 Poverty Traps, Cycles, etc.

- Discuss the case of a general non-concave or non-monotonic G.
- Multiple steady states; unstable versus stable ones; poverty traps.
- Local versus global stability; local convergence rate.
- Oscillating dynamics; perpetual cycles.
- See Figures 6 and 7 for examples.

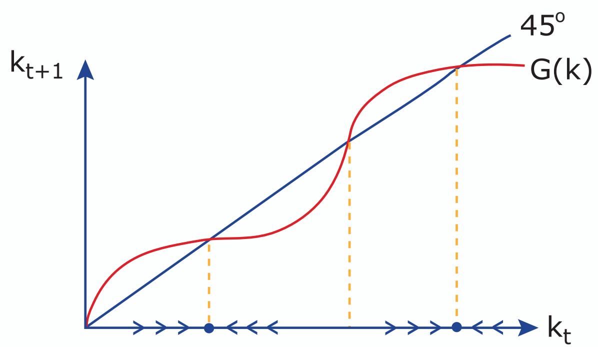


Figure 6

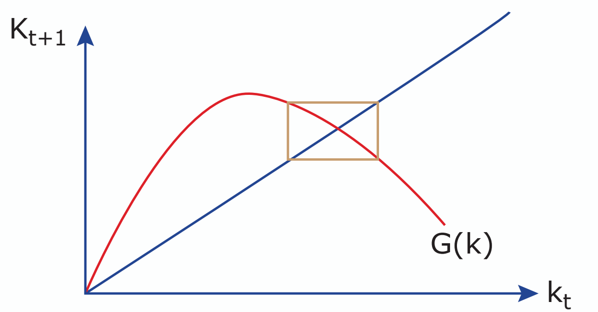


Figure 7

6.7 Introducing Endogenous Growth

- What ensures that the growth rate asymptotes to zero in the Solow model (and the Ramsey model as well) is the vanishing marginal product of capital, that is, the Inada condition $\lim_{k\to\infty} f'(k) = 0$.

- Continue to assume that $f''(k) < 0$, so that $\gamma'(k) < 0$, but assume now that $\lim_{k\to\infty} f'(k) = A > 0$. This implies also $\lim_{k\to\infty} \phi(k) = A$. Then, as $k \to \infty$,

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} \to sA - (n + \delta)$$

- If $sA < (n + \delta)$, then it is like before: The economy converges to k^* such that $\gamma(k^*) = 0$. But if $sA > (n + x + \delta)$, then the economy exhibits diminishing but not vanishing growth: γ_t falls with t, but $\gamma_t \to sA - (n + \delta) > 0$ as $t \to \infty$.

- Consider the special case where $f(k) = Ak$ (linear returns to capital). This is commonly referred as the AK model. The economy then follows a balanced-growth path from the very beginning. Along this path, the growth rate of consumption, output and capital are all equal to $\gamma = sA - (n + \delta)$ in all dates.

- Note then that the growth rate depends both on "preferences" (through s) and on "technology" (through A). Hence, the same forces that determined the long-run level of income in the Solow model now also determine the long-run rate of growth.

- We will later "endogenize" A in terms of policies, institutions, markets, etc.

- For example, Romer/Lucas: If we have human capital or spillover effects,

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha}.$$

  If (for reasons that we will study later) it happens that h is proportional to k, then we get that y is also proportional to k, much alike in the simple AK model that we briefly mentioned earlier (i.e., the version of the Solow model with $f(k) = Ak$).

- In particular, let

$$\tilde{k}_t \equiv k_t + h_t$$

denote the "total" capital of the economy and suppose that there exists a constant $\lambda \in (0,1)$ such that

$$k_t = \lambda \tilde{k}_t \qquad h_t = \lambda \tilde{k}_t.$$

The we can write output as

$$y_t = \tilde{A}_t \tilde{k}_t.$$

where

$$\tilde{A}_t \equiv A_t \lambda^\alpha (1-\lambda)^{1-\alpha}.$$

represent the effective productivity of the "total" capital of the economy. Finally, assuming that households save a fraction s of their income to "total" capital, the growth rate of the economy is then simply

$$\gamma_t = s\tilde{A}_t - (\delta + n).$$

- Clearly, this is the same as in the simple AK model, except for the fact that effective productivity is now endogenous to the allocation of savings between the two types of capital (that is, $\tilde{A}_t$ depends on λ).

- Question: what is the λ that maximizes effective productivity and growth?