Econ 14.04 Fall 2006
Assignment 3: Solutions

1. (a) if we were to replace δ with α, $\frac{1}{1+r}$ with p_1, and c_0, c_1 with x_0, x_1 our problem is identical to:

$$\max_{x_0,x_1} u(x_0,x_1) = \ln(x_0) + \delta \ln(x_1)$$
$$st : x_0 + p_1 x_1 = \omega$$

(b) Taking the FOC we have:

$$\frac{1}{c_0} = \lambda$$
$$\frac{\delta}{c_1} = \lambda \frac{1}{1+r}$$

so:

$$\frac{c_1}{\delta c_0} = (1+r)$$

Substitution into the budget constraint yields:

$$x_0(\delta, r) = \frac{\omega}{1+\delta}$$
$$x_1(\delta, r) = \frac{\delta(1+r)}{1+\delta}$$

(c) when $\delta(1+r) = 1$, then the marshallian demands are equal. In macro the usual condition is phrased: "when the discount rate is equal to the interest rate"

(d) To maximize profits, the agent solves:

$$\max p_x x + p_y y$$
$$st : x^2 + y^2 = 2$$

Note that we do not have to worry about corner solutions because the production possiblity set is a circle and thus when the production set hits the corners, the slopes are zero and infinity respectively. Thus, there is no price vector $\frac{p_1}{p_2}$ that is not tangent to the PPC before hitting the corners. Solving the FOC we get:

$$p_x + \lambda 2x = 0$$
$$p_y + \lambda 2y = 0$$

Thus:

$$\frac{p_x}{p_y} = \frac{x}{y}$$

Plugging these into the production constraint yields:

$$\left(\left(\frac{p_x}{p_y}\right)^2 + 1\right) y^2 = 2$$

$$
\begin{aligned}
x(p_x, p_y) &= \left(\frac{2p_x^2}{p_x^2 + p_y^2}\right)^{1/2} \\
y(p_x, p_y) &= \left(\frac{2p_y^2}{p_x^2 + p_y^2}\right)^{1/2}
\end{aligned}
$$

(e) $p_x = p_y = 1$, thus:

$$x(p_x, p_y) = y(p_x, p_y) = 1$$

We must find the discount rate such that production equals consumption. This will be met if:

$$c_0 = c_1 = 1$$

From part c we know that this must be $\delta = 1$.

2. (a) Setting up the legrangian we have:

$$K : 2x - x^2 + \lambda(K - x)$$

The FOC of this is:

$$
\begin{aligned}
\frac{\partial K}{\partial x} &: \ 2 - 2x - \lambda = 0 \\
\frac{\partial K}{\partial \lambda} &: \ (K - x) \geq 0, \lambda \geq 0, (K - x)\lambda = 0
\end{aligned}
$$

$\lambda(K) = \max((2 - 2K), 0)$. Notice that the slope of $2x - x^2 = 2 - 2x$. In places where the constraint binds $x = K$ and thus the legrangian multiplier is exactly equal to the slope of the function. In economics, λ is often called the "shadow cost." It corresponds to the amount that the agent would be willing to pay to change the constraint slightly. Notice that if the constraint K is binding and is moved slightly from K to $K + \varepsilon$, M would change by:

$$\lim_{\varepsilon \to 0} M(K + \varepsilon) - M(K)$$

As ε goes to zero and we multiply the top and the bottom of this by a little bit we get:

$$\lim_{\varepsilon \to 0} \varepsilon \frac{M(K + \varepsilon) - M(K)}{\varepsilon}$$

The second term in this is simply the slope. Thus

$$\lim_{\varepsilon \to 0} M(K + \varepsilon) - M(K) = \varepsilon \frac{dM}{dK} = \varepsilon\lambda(K)$$

When the constraint is not binding:

$$\lim_{\varepsilon \to 0} M(K + \varepsilon) - M(K) = 0$$

But this is identical to $\lambda\varepsilon$ since $\lambda = 0$. Thus:

$$\lim_{\varepsilon \to 0} M(K + \varepsilon) - M(K) = \varepsilon\lambda(K)$$

In the entire domain.

(b) If we think about Reimann Integration, we will recognize the above difference as exactly one of our rectangles underneath a curve. Thus, to find the rate of change over an area, we are simply going to add up a bunch of rectangles as their widths go to zero. A bit more formally:

$$M(K_1)-M(K_0) = \lim_{N\to\infty} \sum_{i=1}^{N} \left[M(K_0 + (K_1 - K_0)\frac{i+1}{N}) - M(K_0 + (K_1 - K_0)\frac{i}{N}) \right]$$

Notice that as N gets bugger the difference between $\frac{i}{N}$ and $\frac{i+1}{N}$ gets smaller and smaller. Eventually we get something that looks identical to the above:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left[M(K + \frac{i+1}{N}) - M(K + \frac{i}{N}) \right] = \int_{K_0}^{K_1} \lambda(K)dK$$

We could get the same thing using the envelope thm, Recall that the envelope thm says that we can ignore the marginal effects of K on the optimal x to find the change of M with a change in K. Thus:

$$\frac{dM(x(K), K)}{dK} = \lambda$$

In order to find the change over a larger area we simply add up all the marginal changes:

$$M(K_1) - M(K_0) = \int \frac{dM(x(K), K)}{dK} dK = \int \lambda(K)$$

(c) Setting up the FOC:

$$K : \alpha \ln(x) + (1 - \alpha) \ln(y) + \lambda(m - p_x x + p_y y))$$

$$\begin{aligned} \frac{\partial K}{\partial x} &: \frac{\alpha}{x} = \lambda p_x \\ \frac{\partial K}{\partial x} &: \frac{1-\alpha}{y} = \lambda p_y \end{aligned}$$

Thus:

$$\frac{\alpha}{1-\alpha}\frac{y}{x} = \frac{p_x}{p_y}$$

and:

$$\begin{aligned} y(p_x, p_y, m) &= \frac{(1-\alpha)m}{p_y} \\ x(p_x, p_y, m) &= \frac{\alpha m}{p_x} \\ \lambda(\alpha, p_x, m) &= \frac{\alpha}{\frac{\alpha m}{p_x} p_x} = \frac{1}{m} \end{aligned}$$

(d) 1. $\lambda(a) = \frac{1}{m}$

2. $\frac{\partial v}{\partial m} = \lambda(a)$. Thus:

$$v(p_x, p_y, m^{**}) - v(p_x, p_y, m^*) = \int_{m^*}^{m^{**}} \frac{1}{m} dm = \ln(m^{**}) - \ln(m^*)$$

You could check this by plugging everything into the indirect utiltiy function and noting that the indirect utility function has $\ln(m)$ as a seperate term.

3. $v(p_x^{**}, p_y, m) - v(p_x^*, p_y, m) = -\int_{p_x^*}^{p_x^{**}} \frac{1}{m} x^* dm = -\int_{m^*}^{m^{**}} \frac{\alpha}{p_x} dm = \alpha \ln(p_x^*) - \alpha \ln(p_x^{**})$

3. (a) vNM utility functions are unique up to an affine transformation which has two degrees of freedom. For convenience, I will use the convention that $u(A) = 1, u(D) = 0$. From the text:

$$u(B) \tilde{} pu(A) + (1-p)u(D) = p$$

Also from the text:

$$u(C) = qU(b) + (1-q)U(D) = pq$$

Thus a utility function of the expected utitliy form that represents these preferences has:

$$\begin{aligned} u(A) &= 1 \\ u(B) &= p \\ u(C) &= pq \\ u(D) &= 0 \\ U(L) &= \sum_{i \in \{A,B,C,D\}} \lambda_i u(i), \sum_{i \in \{A,B,C,D\}} \lambda_i = 1 \end{aligned}$$

(b) To judge a criterion we look at the probability of the four cases:
Criterion 1:

$$\begin{aligned} \Pr(\text{No Evacuation Nec\&None Performed}) &= \Pr(A) = \underbrace{.99}_{No\ Flood} * \underbrace{.9}_{Evac|NoFlood} = .891 \\ \Pr(\text{No Evacuation Nec\&Performed}) &= \Pr(B) = .99 * .1 = .099 \\ \Pr(\text{Evacuation Nec\&Performed}) &= Pr(C) = .01 * .9 = .009 \\ \Pr(\text{Evacuation Nec\&None Performed}) &= Pr(C) = .01 * .1 = .001 \end{aligned}$$

Thus the total utility in this case is:

$$.891U(A) + .099U(B) + .009U(C) + .001U(D) = .99(.9 + .1p) + .01(.9pq)$$

Criterion 2:

$$(.99 * .95)U(A) + (.99 * .05)U(B) + (.01 * .95)U(C) = .99(.95 + .05p) + .01(.95pq)$$

Subtracting Criterion 2 from Criterion 1 yields:

$$.99(.05 - .05p) + .01(.05pq)$$

Since p$\in (0,1)$ criterion 2 is strictly preferred.

4. (a) This is a simple result of the envelope theorem:

$$\pi(p) = \sum_{y \in Y} p \cdot y$$

From the FOC:

$$p_i + \sum_{j \neq i} p_j \frac{\partial y_j}{\partial y_i} = 0$$

So:

$$\sum_{y \in Y} p \cdot y(p) = y(p) + \frac{\partial y_i}{\partial p_i} \underbrace{\left[\sum_{j \neq i} p_j \frac{\partial y_j}{\partial y_i} + p_i \right]}_{0} = y(p)$$

(b) Ignoring shut down for a minute, the firm that is forced to produce would maximize:

$$p \ln(x) - wx$$

The FOC is:

$$\frac{p}{x} = w \rightarrow x = \frac{p}{w}$$

Plugging these into the maximand we have:

$$\pi(p, w) = p \ln(\frac{p}{w}) - p = p \left[\ln(\frac{p}{w}) - 1 \right]$$

This is only positive when $\ln(\frac{p}{w}) > 1$. Thus:

$$\begin{aligned} x &= \begin{matrix} \frac{p}{w} & \ln(\frac{p}{w}) > 1 \\ 0 & otherwise \end{matrix} \\ \pi(p, w) &= \begin{matrix} p \left[\ln(\frac{p}{w}) - 1 \right] & \ln(\frac{p}{w}) > 1 \\ 0 & otherwise \end{matrix} \end{aligned}$$

5. (a) We see that the isoquant will always have a kink at $x_1 = x_2$. Thus a profit maximizing firm will either choose to not produce ($x_1 = x_2$), or use the same ratio of factor one and two

(b) plugging in $x_1 = x_2$:

$$\max p x_1^\alpha - (w_1 + w_2) x_1$$

FOC:

$$p \alpha x_1^{\alpha - 1} = (w_1 + w_2)$$

Thus:

$$\begin{aligned} x_1 &= x_2 = \left(\frac{(w_1 + w_2)}{p\alpha} \right)^{\frac{1}{\alpha - 1}} = \left(\frac{p\alpha}{(w_1 + w_2)} \right)^{\frac{1}{1 - \alpha}} \\ y(p, w_1, w_2, \alpha) &= \left(\frac{p\alpha}{(w_1 + w_2)} \right)^{\frac{\alpha}{1 - \alpha}} \\ \pi(p, w_1, w_2, \alpha) &= p \left(\frac{p\alpha}{(w_1 + w_2)} \right)^{\frac{\alpha}{1 - \alpha}} - w \left(\frac{(w_1 + w_2)}{p\alpha} \right)^{\frac{1}{\alpha - 1}} \end{aligned}$$

The second derivative is negative iff $\alpha < 1$. Otherwise the production function is CRTS or IRTS and $x_1 = x_2 = \infty$

(c) 1. See ii

2.

$$\min w_1x_1 + w_2x_2$$
$$ST \quad : \quad x_1^\alpha = x_2^\alpha = y$$

We don't need to take the FOC, simply note that for a given y:

$$\begin{aligned} x_1(w_1, w_2, \alpha, y) &= x_2(w_1, w_2, \alpha, y) = y^{\frac{1}{\alpha}} \\ c(w_1, w_2, \alpha, y) &= (w_1 + w_2)y^{\frac{1}{\alpha}} \end{aligned}$$

3. Now solving:

$$\max py - (w_1 + w_2)y^{\frac{1}{\alpha}}$$

We get:

$$\frac{1}{\alpha}y^{\frac{1-\alpha}{\alpha}} = \frac{p}{w_1 + w_2} \to y = \left(\frac{\alpha p}{w_1 + w_2}\right)^{\frac{\alpha}{1-\alpha}}$$