Lecture 4 - Utility Maximization

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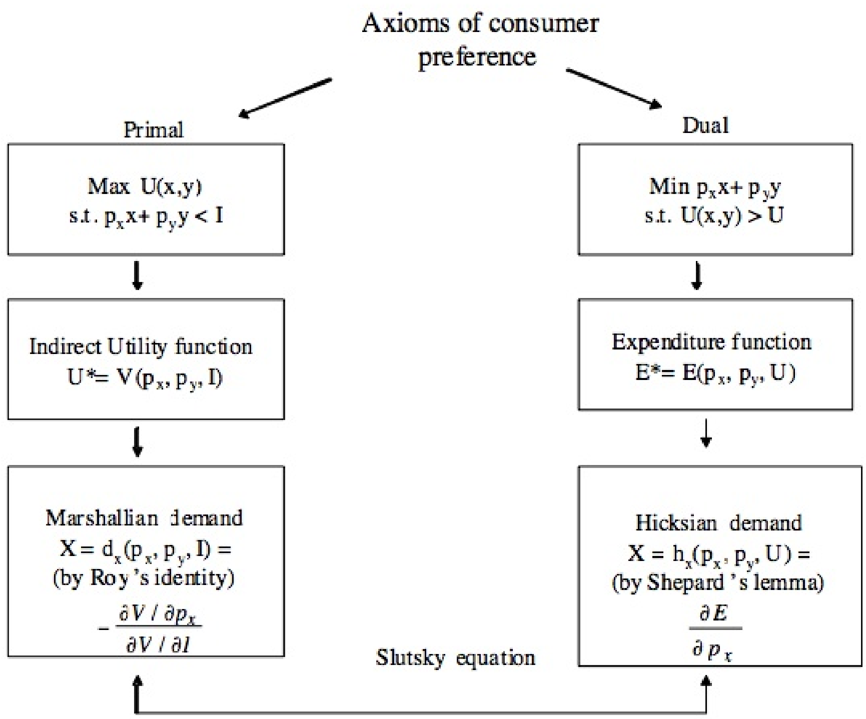
1 Roadmap: Theory of consumer choice

This figure shows you each of the building blocks of consumer theory that we'll explore in the next few lectures. This entire apparatus stands entirely on the five axioms of consumer theory that we laid out in Lecture Note 3. It is an amazing edifice, when you think about it.



2 Utility maximization subject to budget constraint

Ingredients

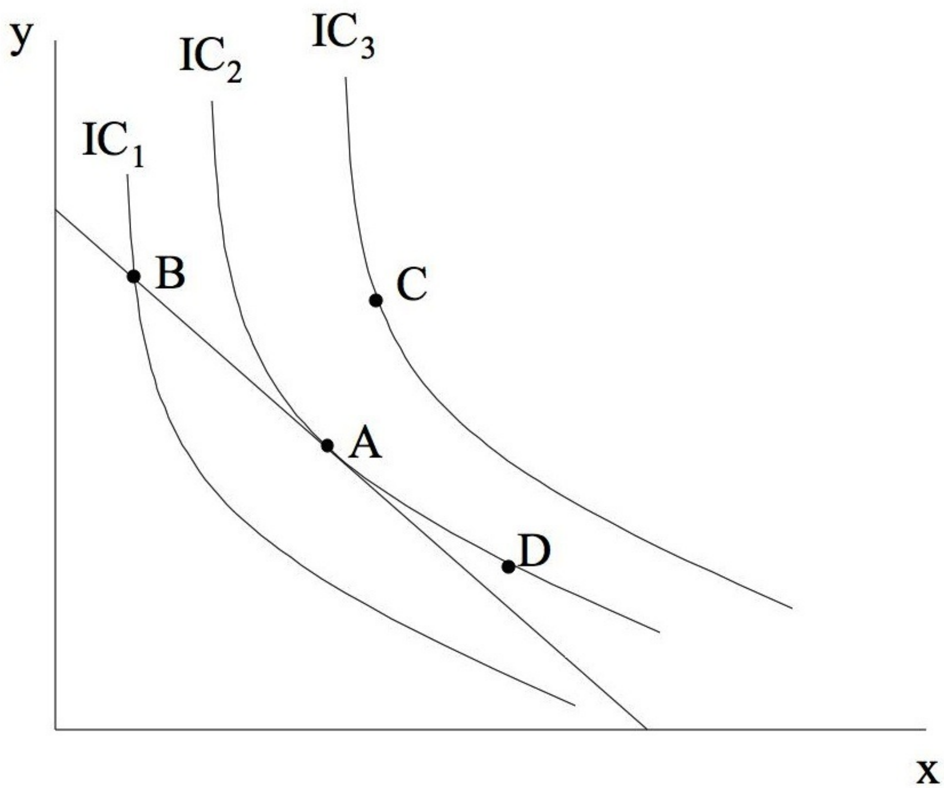
- Utility function (preferences)
- Budget constraint
- Price vector

Consumer's problem

- Maximize utility subject to budget constraint.

- Characteristics of solution:
  - Budget exhaustion (non-satiation)
  - For most solutions: psychic trade-off = market trade-off
  - Psychic trade-off is MRS
  - Market trade-off is the price ratio

- From a visual point of view utility maximization corresponds to point A in the diagram below
  - The slope of the budget set is equal to $-\frac{p_x}{p_y}$
  - The slope of each indifference curve is given by the MRS at that point

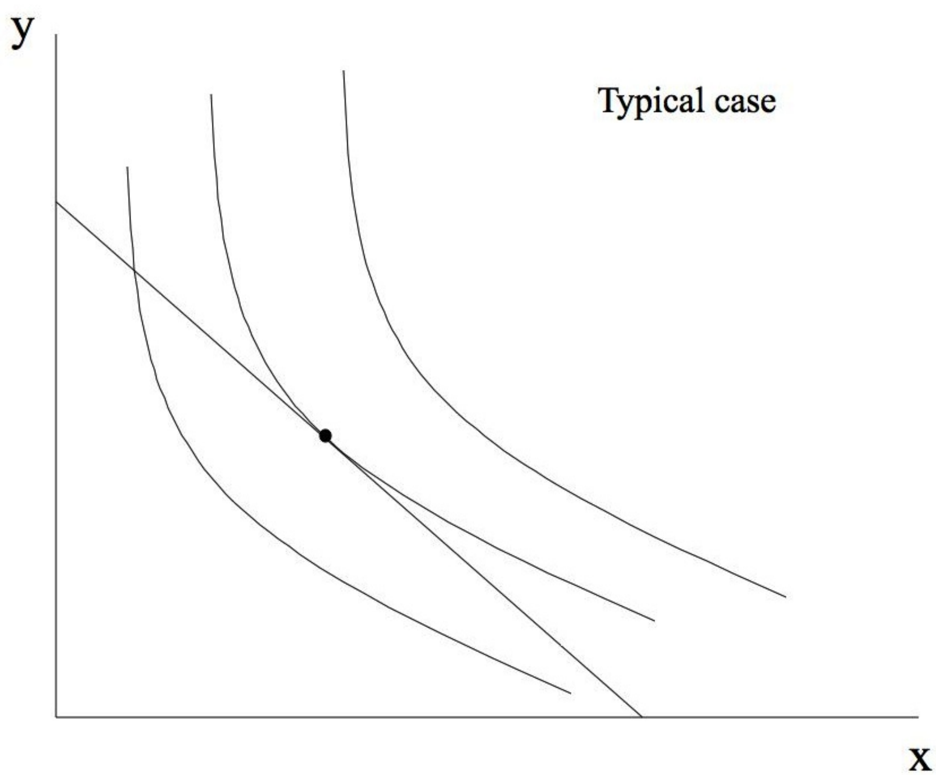


- We can see that $A\ ^P\ B$, $A\ ^I\ D$, $C\ ^P\ A$. Why might we expect someone to choose A?

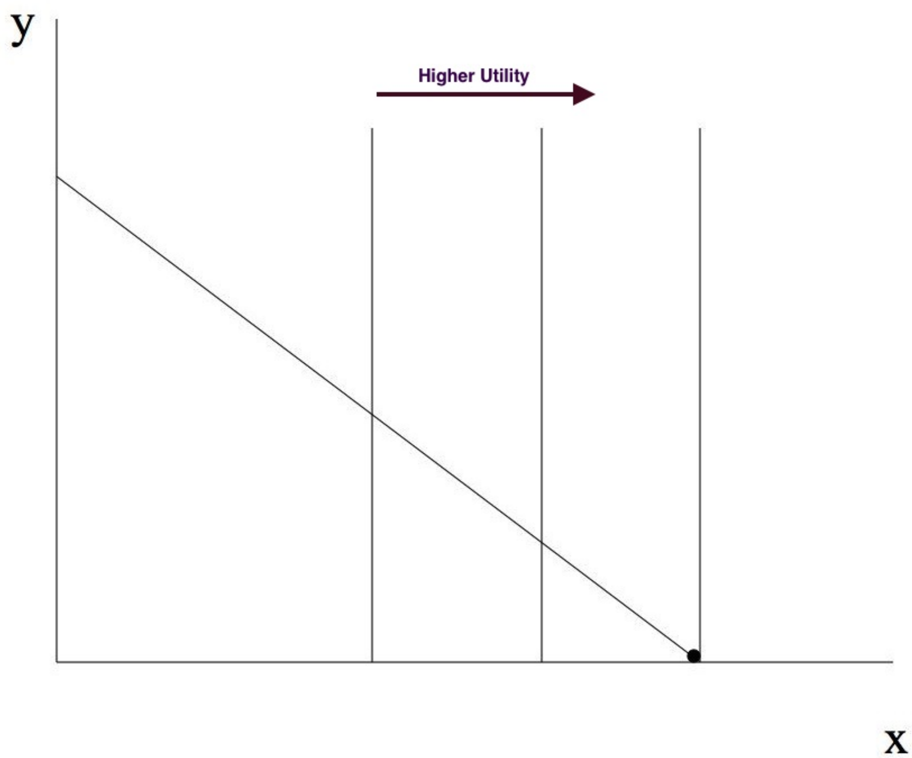
2.1 Interior and corner solutions

There are two types of solution to this problem, interior solutions and corner solutions

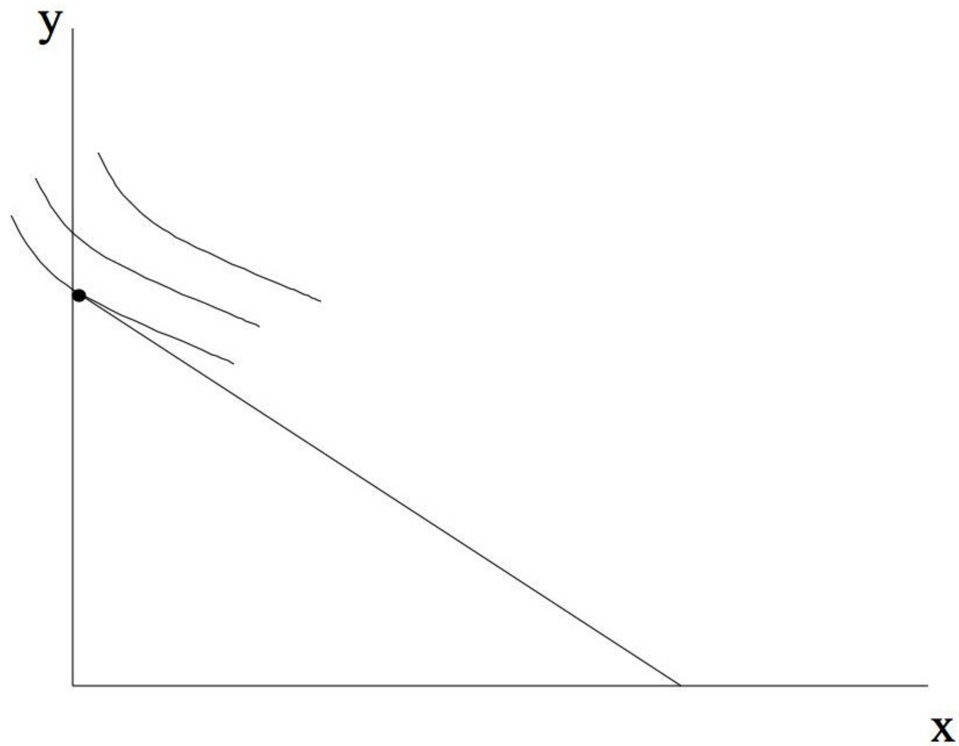
- The figure below depicts an interior solution



- The next figure depicts a corner solution. In this specific example the shape of the indifference curves means that the consumer is indifferent to the consumption of good y. Utility increases only with consumption of x. Thus, the consumer purchases x exclusively.

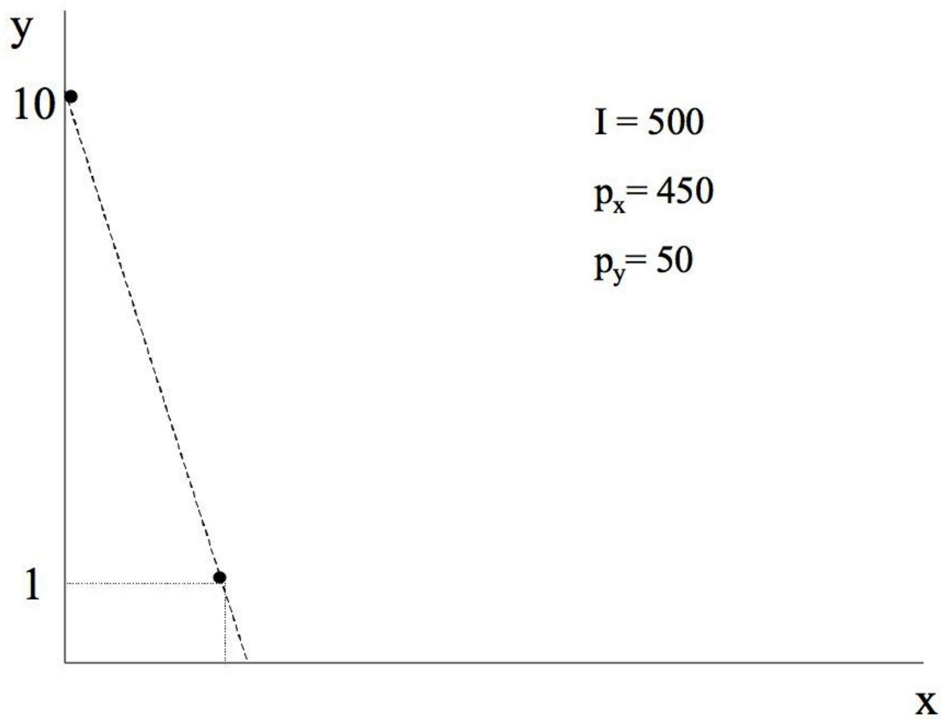


- In the following figure, the consumer's preference for y is sufficiently strong relative to x that the the psychic trade-off is always lower than the monetary trade-off. (This must be the case for many products that we don't buy.)



- What this means is that the corners (more precisely, the axes), serve as constraints. The consumer would prefer to choose a bundle with negative quantities of x and positive quantities of y. That's not feasible in the real world. So to solve the problem using the Lagrangian method, we impose these non-negativity constraints to prevent a non-sensical solution.

- Another type of "corner" solution can result from indivisibilities the bundle (often called integer constraints).



- Given the budget and set of prices, only two bundles are feasible—unless the consumer could purchase non-integer quantities of good x. We usually abstract from indivisibility.

- Going back to the general case, how do we know a solution exists for consumer, i. e. how do we know the consumer would choose a unique bundle? The axiom of completeness guarantees this. Every bundle is on some indifference curve and can therefore be ranked. On page 3 for example: $A \ ^I \ B$, $A \succ B$, $B \succ A$.

2.2 Mathematical solution to the Consumer's Problem

- Mathematics

$$
\begin{aligned}
& \max_{x,y} U(x,y) \\
s.t. \ \ p_x x + p_y y &\leq I \\
\mathcal{L} &= U(x,y) + \lambda(I - p_x x - p_y y) \\
1. \quad \frac{\partial \mathcal{L}}{\partial x} &= U_x - \lambda p_x = 0 \\
2. \quad \frac{\partial \mathcal{L}}{\partial y} &= U_y - \lambda p_y = 0 \\
3. \quad \frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0
\end{aligned}
$$

- Rearranging (1) and (2):

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

This means that the psychic trade-off is equal to the monetary trade-off between the two goods.

- Equation (3) states that budget is exhausted (non-satiation).

- Also notice that:

$$\begin{aligned} \frac{U_x}{p_x} &= \lambda \\ \frac{U_y}{p_y} &= \lambda \end{aligned}$$

- What is the meaning of λ?

2.3 Interpretation of λ, the Lagrange multiplier

- At the solution of the Consumer's problem (more specifically, an interior solution), the following conditions will hold:

$$\frac{\partial U/\partial x}{p_x} = \frac{\partial U/\partial y}{p_y} = \frac{\partial U/\partial x_n}{p_n} = \lambda,$$

and for many goods $(x_1, x_2, ..., x_n)$:

$$\frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = ... = \frac{\partial U/\partial x_n}{p_n} = \lambda$$

This expression says that at the utility-maximizing point, the next dollar spent on each good yields the same marginal utility.

- So what about $\frac{dU(x^*,y^*)}{dI}$, where x^* and y^* are the consumer's optimal consumption choices subject to her budget constraint?

- What is $\frac{dU^*}{dI}$ in that case, where U^* is $U(x^*, y^*)$? Return to Lagrangian:

$$\begin{aligned}
\mathcal{L} &= U(x,y) + \lambda(I - p_x x - p_y y) \\
\frac{\partial \mathcal{L}}{\partial x} &= U_x - \lambda p_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= U_y - \lambda p_y = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0 \\
\frac{d\mathcal{L}}{dI}\Big|_{x=x^*, y=y^*} &= \frac{\partial \mathcal{L}}{\partial x^*}\frac{\partial x^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial y^*}\frac{\partial y^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial I} \\
&= \left(U_x \frac{\partial x^*}{\partial I} - \lambda p_x \frac{\partial x^*}{\partial I}\right) + \left(U_y \frac{\partial y^*}{\partial I} - \lambda p_y \frac{\partial y^*}{\partial I}\right) + \lambda
\end{aligned}$$

  By substituting $\lambda = \frac{U_x}{p_x}\Big|_{x=x^*}$ and $\lambda = \frac{U_y}{p_y}\Big|_{y=y^*}$, we see that both expressions in parenthesis are zero.

- We conclude that:

$$\frac{d\mathcal{L}}{dI} = \frac{\partial \mathcal{L}}{\partial I} = \lambda$$

  λ equals the "shadow price" of the budget constraint, i.e. it expresses the quantity of utils that could be obtained with the next dollar of consumption. *Note that this expression only holds when $x = x^*$ and $y = y^*$. If x and y were not at their optimal values, then the total derivative of L with respect to I would also include additional cross-partial terms. These cross-partials are zero at $x = x^*$ and $y = y^*$.*

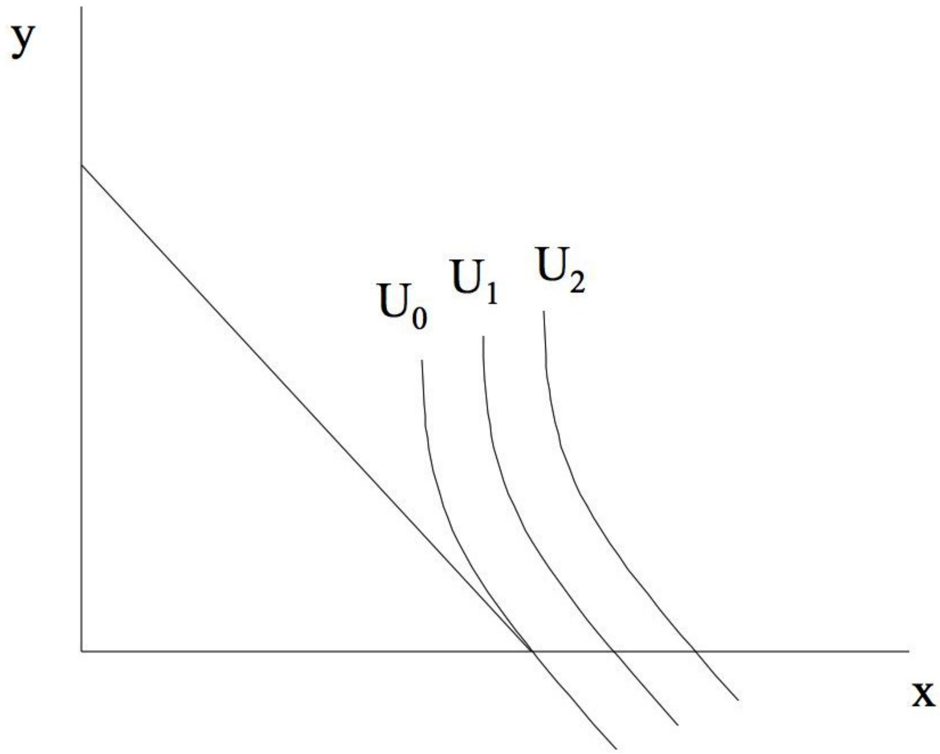
- What does the "shadow price" mean? It's essentially the "utility value" of relaxing the budget constraint by one unit (e.g., one dollar).

- Note that this shadow price is not uniquely defined since it corresponds to the marginal utility of income in "utils," which is an ordinal value. Therefore, the shadow price is defined only up to a monotonic transformation.

- We could also have determined that $d\mathcal{L}/dI = \lambda$ without calculations by applying the envelope theorem. The envelope theorem for constrained problems says that $\frac{dU^*}{dI} = \frac{\partial \mathcal{L}}{\partial I} = \lambda$. Because (at the utility maximizing solution to this problem), x^* and y^* are already optimized, an infinitesimal change in I does not alter these choices. Thus, at x^* and y^*, the effect of I on U depends only on its direct effect on the budget constraint and does not depend on its indirect effect (due to re-optimization) on the choices of x and y. This "envelope" result is only true in a small neighborhood around the solution to the original problem.

2.4 Corner solutions

- When at a corner solution, consumer buys zero of some good and spends the entire budget on other goods.
- What problem does this create for us when we try to solve the Lagrangian?



- The problem above is that a point of tangency doesn't exist for positive values of y. Hence we also need to impose "non-negativity constraints": $x \geq 0$, $y \geq 0$. This will not be important for problems in this class, but it's easy to add these constraints to the maximization problem.

2.5 An Example Problem

- Consider the following example problem:

$$U(x, y) = \frac{1}{4} \ln x + \frac{3}{4} \ln y$$

- Notice that this utility function satisfies all axioms:

  1. Completeness, transitivity, continuity
  2. Non-satiation: $U_x = \frac{1}{4x} > 0$ for all x. $U_y = \frac{3}{4y} > 0$ for all y. In other words, utility rises continually with greater consumption of either good, though the rate at which it rises declines (diminishing marginal utility of consumption).
  3. Diminishing marginal rate of substitution:
     - Along an indifference curve of this utility function: $\bar{U} = \frac{1}{4}\ln x_0 + \frac{3}{4}\ln y_0$.
     - Totally differentiate: $0 = \frac{1}{4x_0}dx + \frac{3}{4y_0}dy$.
     - Which provides the marginal rate of substitution $-\frac{dy}{dx}|_{\bar{U}} = \frac{U_x}{U_y} = \frac{4y_0}{12x_0}$.
     - The marginal rate of substitution of x for y is increasing in the amount of y consumed and decreasing in the amount of x consumed; holding utility constant, the more y the consumer has, the more y she would give up for one additional unit of x.

- Example values: $p_x = 1$, $p_y = 2$, $I = 12$. Write the Lagrangian for this utility function given prices and income:

$$\max_{x,y} U(x,y) = \frac{1}{4}\ln x + \frac{3}{4}\ln y$$

$$\begin{aligned} s.t.\ \ p_x x + p_y y &\leq I \\ \mathcal{L} &= \frac{1}{4}\ln x + \frac{3}{4}\ln y + \lambda(12 - x - 2y) \\ 1.\qquad \frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{4x} - \lambda = 0 \\ 2.\qquad \frac{\partial \mathcal{L}}{\partial y} &= \frac{3}{4y} - 2\lambda = 0 \\ 3.\qquad \frac{\partial \mathcal{L}}{\partial \lambda} &= 12 - x - 2y = 0 \end{aligned}$$

- Rearranging (1) and (2), we have

$$\begin{aligned} \frac{U_x}{U_y} &= \frac{p_x}{p_y} \\ \frac{1/4x}{3/4y} = \frac{x}{3y} &= \frac{1}{2} \end{aligned}$$

- The interpretation of this expression is that the MRS (psychic trade-off) is equal to the market trade-off (price-ratio).

- What's $\frac{d\mathcal{L}}{dI}$? As before, this is equal to λ, which from (1) and (2) is equal to:

$$\lambda = \frac{1}{4x^*} = \frac{3}{8y^*}.$$

The next dollar of income could buy one additional x, which has marginal utility $\frac{1}{4x^*}$ or it could buy $\frac{1}{2}$ additional y, which provides marginal utility $\frac{3}{4y^*}$ (so, the marginal utility increment is $\frac{1}{2} \cdot \frac{3}{4y^*}$).

- It's important that $dL/dI = \lambda$ is defined in terms of the optimally chosen x^*, y^* . Unless we are at these optimal points, the envelope theorem does not apply. In that case, $d\mathcal{L}/dI$ would also depend on the cross-partial terms: $\left(U_x \frac{\partial x}{\partial I} - \lambda p_x \frac{\partial x}{\partial I}\right) + \left(U_y \frac{\partial y}{\partial I} - \lambda p_y \frac{\partial y}{\partial I}\right)$.

- Incidentally, you should be able to solve for the prices and budget given, $x^* = 3, y^* = 4.5$.

- Having solved that, you can verify that $\frac{1}{4x^*} = \frac{3}{8y^*} = \lambda$. That is, at prices $p_x = 1$ and $p_y = 2$ and consumption choices $x^* = 3, y^* = 4.5$, the marginal utility of a dollar spent on either good x or good y is identical.

2.6 Lagrangian with Non-negativity Constraints [Optional]

$$\begin{aligned}
& \max U(x,y) \\
s.t.\ p_x x + p_y y &\leq I \\
y &\geq 0 \\
\mathcal{L} &= U(x,y) + \lambda(I - p_x x - p_y y) + \mu\,(y - 0) \\
\frac{\partial \mathcal{L}}{\partial x} &= U_x - \lambda p_x = 0 \\
\partial \mathcal{L} &= U_y - \lambda p_y + \mu = 0 \\
\mu y &= 0
\end{aligned}$$

- Final equation above implies that $\mu = 0$, $y = 0$, or both. (This is called a "complementary slackness" condition: either the constraint is slack, implying $\mu = 0$, or the constraint is binding, implying that $y = 0$, and so in either case, we have that the product $\mu y = 0$.)

- We then have three cases.

1. $y = 0,\ \mu \neq 0$ (since $\mu \geq 0$ then it must be the case that $\mu > 0$)

$$\begin{aligned} U_y - \lambda p_y + \mu &= 0 \longrightarrow U_y - \lambda p_y < 0 \\ \frac{U_y}{p_y} &< \lambda \\ \frac{U_x}{p_x} &= \lambda \end{aligned}$$

Combining the last two expressions:

$$\frac{U_x}{U_y} > \frac{p_x}{p_y}$$

This consumer would like to consume even more x and less y, but she cannot.

2. $y \neq 0,\ \mu = 0$

$$\begin{aligned} U_y - \lambda p_y + \mu &= 0 \longrightarrow U_y - \lambda p_y = 0 \\ \frac{U_y}{p_y} &= \frac{U_x}{p_x} = \lambda \end{aligned}$$

Standard FOC, here the non-negativity constraint is not binding.

3. $y = 0,\ \mu = 0$

Same FOC as before:

$$\frac{p_x}{p_y} = \frac{U_x}{U_y}$$

Here the non-negativity constraint is satisfied with equality so it doesn't distort consumption.

3 Indirect Utility Function

- For any:
  - Budget constraint
  - Utility function
  - Set of prices

We obtain a set of optimally chosen quantities:

$$\begin{aligned} x_1^* &= x_1(p_1, p_2, ..., p_n, I) \\ &... \\ x_n^* &= x_n(p_1, p_2, ..., p_n, I) \end{aligned}$$

So when we say

$$\max U(x_1, ..., x_n) \; s.t. \; p_1 x_1 + ... + p_n x_n \leq I$$

we get as a result:

$$U(x_1^*(p_1, ..., p_n, I), ..., x_n^*(p_1, ..., p_n, I)) \equiv V(p_1, ..., p_n, I).$$

We call $V(\cdot)$ the "Indirect Utility Function." This is the value of maximized utility under given prices and income.

- So remember the distinction:
  - Direct utility: utility from consumption of $(x_1, ..., x_n)$
  - Indirect utility: utility obtained when facing the set of prices and income given by $(p_1, ..., p_n, I)$

- Example

$$\begin{aligned} \max U(x, y) &= x^{.5} y^{.5} \\ s.t. \; p_x x + p_y y &\leq I \\ \mathcal{L} &= x^{.5} y^{.5} + \lambda(I - p_x x - p_y y) \\ \frac{\partial \mathcal{L}}{\partial x} &= .5 x^{-.5} y^{.5} - \lambda p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= .5 x^{.5} y^{-.5} - \lambda p_y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0 \end{aligned}$$

- We obtain the following:

$$\lambda = \frac{.5 x^{-.5} y^{.5}}{p_x} = \frac{.5 x^{.5} y^{-.5}}{p_y},$$

which simplifies to:

$$x = \frac{p_y y}{p_x}.$$

- Substituting into the budget constraint gives us

$$\begin{aligned} I - p_x \frac{p_y y}{p_x} - p_y y &= 0 \\ p_y y &= \frac{1}{2} I, \quad p_x x = \frac{1}{2} I \end{aligned}$$

$$x^* = \frac{I}{2p_x}, \quad y^* = \frac{I}{2p_y}$$

Half of the budget goes to each good.

- Thus, for a consumer with $U(x, y) = x^{0.5} y^{0.5}$, budget I, and facing prices p_x and p_y will choose x^* and y^* and obtain utility:

$$U(x^*, y^*) = \left(\frac{I}{2p_x}\right)^{.5} \left(\frac{I}{2p_y}\right)^{.5}.$$

Thus, the indirect utility for this consumer is

$$V(p_x, p_y, I) = U(x^*(p_x, p_y, I), y^*(p_x, p_y, I)) = \left(\frac{I}{2p_x}\right)^{.5} \left(\frac{I}{2p_y}\right)^{.5}$$

- Why bother calculating the indirect utility function? It saves us time. Instead of recalculating the utility level for every set of prices and budget constraints, we can plug in prices and income to get consumer utility. This comes in handy when working with individual demand functions. Demand functions give the quantity of goods purchased by a given consumer as a function of prices and income.