Lecture 18

Ocean General Circulation Modeling

9.1 The equations of motion: Navier-Stokes

The governing equations for a real fluid are the Navier-Stokes equations (conservation of linear momentum and mass mass) along with conservation of salt, conservation of heat (the first law of thermodynamics) and an equation of state. However, these equations support fast acoustic modes and involve nonlinearities in many terms that makes solving them both difficult and expensive and particularly ill suited for long time scale calculations. Instead we make a series of approximations to simplify the Navier-Stokes equations to yield the "primitive equations" which are the basis of most general circulations models.

In a rotating frame of reference and in the absence of sources and sinks of mass or salt the Navier-Stokes equations are

$$\partial_t \rho\vec{v} + \nabla\cdot\rho\vec{v}\vec{v} + 2\vec{\Omega}\wedge\rho\vec{v} + g\rho\hat{k} + \nabla p = \nabla\cdot\vec{\tau} \tag{9.1}$$

$$\partial_t \rho + \nabla\cdot\rho\vec{v} = 0 \tag{9.2}$$

$$\partial_t \rho S + \nabla\cdot\rho S\vec{v} = 0 \tag{9.3}$$

$$\partial_t \rho\theta + \nabla\cdot\rho\theta\vec{v} = \frac{1}{c_{pS}}\nabla\cdot\mathcal{F}_\theta \tag{9.4}$$

$$\rho = \rho(\theta, S, p) \tag{9.5}$$

Where ρ is the fluid density, $\vec{v}$ is the velocity, p is the pressure, S is the salinity and θ is the potential temperature which add up to seven dependent variables.

The constants are $\vec{\Omega}$ the rotation vector of the sphere, g the gravitational acceleration and c_p the specific heat capacity at constant pressure. $\vec{\tau}$ is the stress tensor and $\mathcal{F}_\theta$ are non-advective heat fluxes (such as heat exchange across the sea-surface).

9.2 Acoustic modes

Notice that there is no prognostic equation for pressure, p, but there are two equations for density, ρ; one prognostic and one diagnostic. We can obtain a prognostic equation for pressure by re-writing the continuity equation as

$$D_t\rho = -\rho\nabla\cdot\vec{v} \tag{9.6}$$

and using the chain rule on the equation of state

$$D_t\rho = \left.\frac{\partial\rho}{\partial\theta}\right|_{S,p} D_t\theta + \left.\frac{\partial\rho}{\partial S}\right|_{\theta,p} D_tS + \left.\frac{\partial\rho}{\partial p}\right|_{\theta,S} D_tp. \tag{9.7}$$

We can now eliminate $D_t\rho$ so that for adiabatic motion we get

$$\frac{1}{c_s^2}D_tp = -\rho\nabla\cdot\vec{v} + \rho\alpha D_t\theta - \rho\beta D_tS \tag{9.8}$$

where $c_s = c_s(\theta, S, p)$ is the speed of sound, $\alpha = \alpha(\theta, S, p)$ is the thermal expansion coefficient and $\beta = \beta(\theta, S, p)$ is the haline expansion coefficient.

The linear form of this equation with the linearized momentum equation describes sound waves:

$$\begin{aligned}\bar{\rho}\partial_t\vec{v} &= -\nabla p\\ \partial_tp &= -\bar{\rho}c_s^2\nabla\cdot\vec{v}\end{aligned}$$

or

$$\partial_{tt}p = c_s^2\nabla^2p$$

The speed of sound in water is of the order $c_s \sim 1500$ m s^{-1}. In one minute sound travels approximately 100 km. Even for coarse resolution global models, say with a 300 km grid spacing, stability criteria would limit the time-step to the order of minutes which is unpractical for climate scale calculations. It is therefore necessary to "filter" the equations, removing the acoustic modes as natural modes of the system. Acoustic modes can be filtered by removing the density dependence on pressure or by making the hydrostatic approximation. We will use the first in conjunction with the Boussinesq approximation.

9.3 The Boussinesq approximation

The first simplifying approximation we make is the Boussinesq approximation. It essentially uses the fact that dynamic perturbations in density, ρ', are small compared to the background mean, $\bar{\rho}$:

$$\rho' << \bar{\rho} \tag{9.9}$$

The Boussinesq approximation allows us to linearize terms involving a product with density (e.g. $\rho\vec{v} \rightarrow \bar{\rho}\vec{v}$). The only term which is not a product is the gravitational acceleration term, $g\rho$, and so this term is already linear and is unaffected by the Boussinesq approximation. In many applications this is done with no consideration of the equation of state and can lead to some inconsistencies in the resulting equations. Here we will make this linearization in conjunction with a change in the equation of state as follows. We will let the speed of sound become infinite, $c_s \rightarrow \infty$ but account for dependencies on depth so that the equation of state becomes

$$\rho = \rho(\theta, S, \bar{p}(z)).$$

Using the chain rule of this in 9.6 gives a replacement for 9.8 which under adiabatic conditions is

$$\nabla \cdot \vec{v} = 0.$$

Note that if we don't make the adiabatic assumption the continuity equation should be $\nabla \cdot \vec{v} = \alpha D_t \theta - \beta D_t S$. It is still the case that compressibility terms (on the right hand side) are small compared to any of the three terms on the left hand side (i.e. $\alpha D_t \theta << \partial_z w$). These compressibility terms have recently been the subject of concern and are being added back into models in various ways. However, for the purposes of these lectures notes we will assume that the flow is non-divergent.

With these two assumptions (the replacement of ρ with $\bar{\rho}$ and the non-divergence of the flow) we can write down the non-hydrostatic Boussinesq equations:

$$D_t\vec{v} + 2\vec{\Omega} \wedge \vec{v} + \frac{g\rho}{\bar{\rho}}\hat{k} + \frac{1}{\bar{\rho}}\nabla p = \frac{1}{\bar{\rho}}\nabla \cdot \vec{\tau} \tag{9.10}$$

$$\nabla \cdot \vec{v} = 0 \tag{9.11}$$

$$D_t S = 0 \tag{9.12}$$

$$D_t\theta = \frac{1}{\bar{\rho}c_{pS}}\nabla\cdot\mathcal{F}_\theta \quad (9.13)$$

$$\rho = \rho(\theta, S, z) \quad (9.14)$$

These equations do not permit acoustic modes but nevertheless are expensive to solve for the following reason. The equations are prognostic in the three components of velocity but at the same time the velocity must be constrained to be non-divergent. Solving these simultaneous equations involves inverting a three dimensional elliptic equation as will be found in 9.4. We can, however, simplify the equations further by assuming hydrostatic balance in the vertical which we'll discuss later.

9.3.1 Comments on the derivation of Boussinesq equations

In deriving the Boussinesq equations, we have taken a less conventional route than in many texts. This is in the hope of avoiding some common points of confusion. Even though we have tried to be careful here we should point where the pitfalls are.

One common mistake is to assume when making the non-divergence approximation, the conservation of mass can consequently be split into two equations:

$$\left.\begin{array}{l} D_t\rho + \rho\nabla\cdot\vec{v} = 0 \\ \\ \nabla\cdot\vec{v} = 0 \end{array}\right\} \not\Rightarrow D_t\rho = 0$$

This is because the scaling tells us that $D_t\rho << (\partial_x u, \partial_y v, \partial_z w)$ so that $\nabla\cdot\vec{v} \approx 0$ is the leading order approximation to continuity. At the next order, $D_t\rho$ is balanced by the residual of $\nabla\cdot\vec{v}$ (the divergent part of the flow).

We should also say that the phrase "Boussinesq approximation" is often used to mean any part of the above derivation or all of the above. The most common usage is in the linearization of the momentum equations.

Finally, the dependence of density and expansion coefficients on pressure is actually crucial for getting relative densities of water masses correct. For this reason, ocean modelers either put back the pressure dependence or make the coefficients a function of depth, z:

$$\rho = \rho(\theta, S, \bar{p}(z))$$

In either case consistency of the approximations is broken. In particular, higher moment equations (namely energy conservation) is compromised. It seems that to retain higher momentum properties of the equations under the Boussinesq approximation the equation of state must not only be independent of pressure but be linear in θ and S.

9.4 The pressure method

As mentioned above, there is no explicit equation for pressure and so one must be derived. First, we re-write the momentum equations in a succinct form

$$\partial_t \vec{v} + \frac{1}{\bar{\rho}} \nabla p = \vec{G}$$

where all other terms have been included in $\vec{G}$. Now discretize the time-derivative

$$\vec{v}^{n+1} + \frac{\Delta t}{\bar{\rho}} \nabla p = \vec{v}^n + \Delta t \vec{G}$$

and substitute into the continuity equation at $t = (n+1)\Delta t$

$$\nabla \cdot \vec{v}^{n+1} = 0$$

giving

$$\frac{\Delta t}{\bar{\rho}} \nabla^2 p = \nabla \cdot \vec{v}^n + \Delta t \nabla \cdot \vec{G} \tag{9.15}$$

Solving 9.15 ensures that the future flow, $\vec{v}^{n+1}$, will be non-divergent. The equation is a three dimensional elliptic equation as so quite costly to solve. For flat bottom domains, a modal decomposition in the vertical can be used to render N two-dimensional elliptic problems. If the domain is a box, channel or doubly periodic then direct methods such as a Fourier transformation can be used since the operator has constant coefficients.

One advantage of the pressure method is that it works very well in irregular domains. At boundaries, the no normal flow condition leads to a homogeneous Neumann condition ($\nabla p \cdot \hat{n} = \vec{G} \cdot \hat{n} = 0$. However, in irregular domains the elliptic problem is harder to solve and is normally done so iteratively.

9.5 The elimination method

An alternative to the pressure method is often used in spectral models for process studies. Since there is no explicit equation for pressure, one method of solving the equations is to re-arrange them in order to eliminate pressure. Since the pressure term is a gradient, taking the curl of the equations will eliminate the pressure. For convenience, we'll re-write the Boussinesq momentum equations in vector invariant form:

$$\partial_t \vec{v} + \vec{\omega} \wedge \vec{v} + \frac{g\rho}{\bar{\rho}}\hat{k} + \frac{1}{\bar{\rho}}\nabla(p + \frac{1}{2}\bar{\rho}\vec{v}\cdot\vec{v}) = 0$$

where $\vec{\omega}$ is the absolute vorticity vector

$$\vec{\omega} = 2\vec{\Omega} + \nabla \wedge \vec{v}$$

Taking the curl of these equations gives

$$\partial_t \nabla \wedge \vec{v} + \nabla \wedge (\vec{\omega} \wedge \vec{v}) - \hat{k} \wedge \nabla \frac{g\rho}{\bar{\rho}} = 0 \tag{9.16}$$

This has eliminated pressure but is prognostic in three components of a vector quantity. Taking the curl again gives

$$-\partial_t \nabla^2 \vec{v} + \nabla\left(\nabla \cdot (\vec{\omega} \wedge \vec{v})\right) - \nabla^2(\vec{\omega} \wedge \vec{v}) - \nabla \wedge (\hat{k} \wedge \nabla \frac{g\rho}{\bar{\rho}}) = 0 \tag{9.17}$$

where we have used the identity $\nabla \wedge \nabla \wedge \vec{f} = \nabla\left(\nabla \cdot \vec{f}\right) - \nabla^2 \vec{f}$. Now although 9.17 is prognostic in (u, v, w) and pressure has been eliminated integrating these equations forward would not necessarily satisfy non-divergence. Instead we use only the vertical component of 9.16 and 9.17 and split the horizontal flow into a rotation part, $v_\psi = \hat{k} \wedge \nabla\psi$ and a divergent part, $v_\chi = \nabla\chi$. The equations governing the flow then are

$$\begin{aligned}
\partial_t \nabla_h^2 \psi + \hat{k} \cdot \nabla \wedge (\vec{\omega} \wedge \vec{v}) &= 0 \\
\partial_t \nabla^2 w - \partial_z\left(\nabla \cdot (\vec{\omega} \wedge \vec{v})\right) + \nabla^2\left(\hat{k} \cdot (\vec{\omega} \wedge \vec{v})\right) &= \frac{g}{\bar{\rho}}\nabla_h^2 \rho \\
\nabla_h^2 \chi &= -\partial_z w \\
\vec{v}_h &= \nabla_h \chi + \hat{k} \wedge \nabla\psi
\end{aligned}$$

These equations look more complicated than the primitive equations (which they are) and solving them involves solving two three dimensional elliptic equations and N two dimensional elliptic equations. However, when using the spectral method, inverting such operators is as simple as multiplying by scalars and so is not a concern. The advantage of the method is that there are fewer terms. Note that the flow is driven only by horizontal gradients in buoyancy. This method is only ever used in process studies in spectral codes.

9.6 Hydrostatic balance

The vertical component of the Boussinesq momentum equations is

$$D_t w + 2\Omega\cos\phi v + \frac{g\rho}{\bar{\rho}} + \frac{1}{\bar{\rho}}\partial_z p = \frac{1}{\bar{\rho}}\nabla\cdot\vec{\tau}^w$$

and a scaling for each term reveals that for long horizontal motions ($L >> H$) the dominant balance is

$$\partial_z p = -g\rho \tag{9.18}$$

This allows the pressure to be found by a simple vertical integral. Using

$$p = 0 \quad \text{at} \quad z = \eta$$

where $z = \eta$ is the position of the sea-surface, the internal pressure is then

$$p = \int_z^\eta g\rho \, dz$$

9.7 The free-surface

The integral for pressure starts at the free-surface where the pressure is known (or assumed to be zero). The height of the free-surface is driven up or down by convergence/divergence of the fluid throughout the underlying water column.

We obtain a prognostic equation for free surface height by integrating the non-divergent continuity equation in the vertical from bottom at $z = -H(x,y)$ to top at $z = \eta(x,y,t)$:

$$\int_{-H}^{\eta} \partial_z w \, dz = [w]_{-H}^{\eta} = -\int_{-H}^{\eta} \nabla_h\cdot\vec{v}_h \, dz$$

The right hand side can be re-arranged using Leibniz rule:

$$\int_{-H}^{\eta} \nabla_h\cdot\vec{v}_h \, dz = \nabla_h\cdot\int_{-H}^{\eta}\vec{v}_h \, dz - \vec{v}_h\big|_{z=\eta}\cdot\nabla_h\eta + \vec{v}_h\big|_{z=-H}\cdot\nabla_h(-H)$$

A boundary condition of normal flow into the solid bottom gives a condition on $w(z = -H)$:

$$w|_{z=-H} = -\vec{v}_h \cdot \nabla_h H$$

At the free surface, the interface moves with the fluid so that

$$D_t \eta = w|_{z=\eta} + (P - E)$$

where $E - P$ is excess evaporation over precipitation. The free surface equation then is

$$D_t \eta + \vec{v}_h \cdot \nabla_h H = -\nabla_h \cdot \int_{-H}^{\eta} \vec{v}_h \, dz + \vec{v}_h|_{z=\eta} \cdot \nabla_h \eta - \vec{v}_h|_{z=-H} \cdot \nabla_h(-H) + (P - E)$$

or

$$\partial_t \eta + \nabla_h \cdot \int_{-H}^{\eta} \vec{v}_h \, dz = (P - E)$$

9.8 The hydrostatic primitive equations

The equations now reduce to

$$D_t \vec{v}_h + f\hat{k} \wedge \vec{v}_h + \frac{1}{\bar{\rho}} \nabla_h p = \frac{1}{\bar{\rho}} \nabla \cdot \vec{\tau}_h \qquad (9.19)$$

$$p = \int_z^{\eta} g\rho \, dz \qquad (9.20)$$

$$\partial_z w = -\nabla_h \cdot \vec{v}_h \qquad (9.21)$$

$$D_t S = 0 \qquad (9.22)$$

$$D_t \theta = \frac{1}{\bar{\rho} c_p} \nabla \cdot \mathcal{F}_\theta \qquad (9.23)$$

$$\rho = \rho(\theta, S, z) \qquad (9.24)$$

$$\partial_t \eta = -\nabla_h \cdot \int_{-H}^{\eta} \vec{v}_h \, dz + (P - E) \qquad (9.25)$$

We now have one equation per dependent variable. Three equations are diagnostic, one for each of ρ, p and w. Four prognostic equations describe baroclinic or three-dimensional evolution and correspond to a pair of gravity modes, a geostrophic mode (Rossby wave) and a thermo-haline mode. The free-surface equation couples with the depth integrated momentum equations to give a a pair of external gravity modes and an external Rossby mode.

9.9 Time-step limitations in HPEs

We already saw that sound waves, if permitted, would limit the time step of an explicit large scale model to only the order of minutes. Filtering the equations removed the sound waves and so removes the limitation on time step. The remaining processes in the HPEs are listed in table 9.1, along with their approximate limitation on time-step.

Process	Speed	Formula for maximum Δt	Maximum Δt using Δx= 20 km	Maximum Δt using Δx= 200 km
Sound waves, c_s	1500 m s^{-1}	$\Delta x/c_s$	15 sec	2 min
External waves, $\sqrt{gH}$	200 m s^{-1}	$\Delta x/\sqrt{gH}$	2 min	15 min
Internal waves, NH	3 m s^{-1}	$\Delta x/NH$	2 hour	18 hour
Jets, U	2 m s^{-1}	$\Delta x/U$	3 hour	1 day
Interior flow, U	0.1 m s^{-1}	$\Delta x/U$	2 days	3 weeks

Table 9.1: Approximate limitations on time-step due to propagation processes in the ocean for two resolutions.

9.10 External gravity waves

The next fast process after acoustic waves that can limit the time step are external gravity waves. These motions can be analyzed by depth integrating the equations. If the pressure is split into two parts

$$p = \int_z^\eta g\bar{\rho}\, dz + \int_z^\eta g(\rho - \bar{\rho})\, dz$$

we see that lateral gradients of the first part will be uniform with depth and a function only of the free-surface height:

$$\frac{1}{\bar{\rho}}\nabla_h \int_z^\eta g\bar{\rho}\, dz = g\nabla_h \eta$$

The depth averaged momentum equations and free surface equations can then be summarized as

$$\begin{aligned} \partial_t < \vec{v}_h > + g\nabla_h \eta &= \ldots \\ \partial_t \eta + \nabla \cdot H < \vec{v}_h > &= \ldots \end{aligned}$$

which can be combined to form a wave equation of the form

$$\partial_{tt}\eta - \nabla_h \cdot gH\nabla_h \eta = \ldots$$

which describes waves that propagate with phase/group speed $c_g = \sqrt{gH}$. For a nominal depth of $H = 4$ km, this gives waves speeds of order 200 m s^{-1}. For resolutions of 20 km and 200 km, an explicit time-step would be limited to of order 2 minutes and 15 minutes respectively. This is somewhat smaller than the next explicit frequency in the system (Coriolis, $(2\Omega)^{-1} \sim 7000$ seconds ~ 2 hours).

There are three conventional methods for avoiding this time step limitation; filtering using the rigid-lid approximation, the split-explicit method and the implicit method.

9.11 Rigid-lid approximation

Just as sound waves were filtered out of the equations by removing the time dependency that led to acoustic propagation, we can filter out the explicit gravity waves by appropriately modifying the equations.

In the rigid lid approximation, the surface is approximated as being at $z = 0$. The free-surface equation is modified since the interface can no longer evolve freely, $D_t\eta = 0$:

$$\nabla \cdot \int_{-H}^{0} \vec{v}_h \, dz = 0$$

Next we partition the pressure into a part associated with the rigid-lid, p_s, and the hydrostatic part, p_h:

$$p = p_s + p_h$$

The surface pressure, p_s, is said to be the pressure exerted by the rigid-lid at $z = 0$. The hydrostatic part is found by vertically integrating the hydrostatic equation from $z = 0$ down to some depth z using the boundary condition that $p_h(z = 0) = 0$:

$$p_h = \int_{z}^{0} \frac{g\rho}{\bar{\rho}} \, dz$$

The momentum equations can now be written succinctly as

$$\partial_t \vec{v}_h + \frac{1}{\bar{\rho}} \nabla_h p_s = \vec{G}_h$$

where all other terms including the lateral gradient of hydrostatic pressure are incorporated into $\vec{G}_H$. Discretizing in time

$$\vec{v}_h^{n+1} + \frac{\Delta t}{\bar{\rho}} \nabla_h p_s = \vec{v}_h^n + \Delta t \vec{G}_h$$

and substituting into the depth integrated continuity gives

$$\frac{\Delta t}{\bar{\rho}} \nabla_h \cdot H \nabla_h p_s = \nabla \cdot \int_{-H}^{0} (\vec{v}_h^n + \Delta t \vec{G}_h) \, dz.$$

The algorithm requires calculating $\vec{G}_h$ and then solving this elliptic equation for surface pressure that ensures the depth integrated flow will be non-divergence at the future step. This is known as the pressure method and is exactly the same as the three dimensional pressure method described for solving the non-hydrostatic Boussinesq equations. However, here, there is only a two dimensional elliptic equation to solve which is considerably cheaper.

Alternatively, rather than stepping forward for the full flow, as above, the flow can be split into barotropic (depth integrated) and baroclinic parts:

$$\vec{v}_h =< \vec{v}_h > \vec{v}_h'$$

so that $< \vec{v}_h' >= 0$. Because the depth averaged part is non-divergent it can be written in terms of a stream function, ψ:

$$< u >= -\partial_y \psi \quad ; \quad < v > \partial_x \psi$$

and the curl of the depth integrated momentum equations used to find the depth integrated vorticity

$$< \zeta >= \nabla_h^2 \psi$$

which can be inverted for ψ. We again are solving an 2-D elliptic equation but the boundary conditions require specifying the stream function on coasts ($H = 0$) which is a non-trivial exercise. The stream function method was used in the original ocean model of Cox and Bryan and has the dubious advantage that the elliptic equation does not need to be solved accurately! This is because even a random ψ, let alone a poor estimate, still produces a non-divergent flow by definition. This avoids the computational bottle neck associated with solving the elliptic equation. However, these days, we are can solve elliptic equations quite efficiently and since the pressure method handles irregular domains very well, it is the preferred method.

9.12 Implicit free surface

Yet another alternative is to use an implicit-in-time treatment of the free-surface terms. We saw in the time stepping section (1.3) that implicit methods are unconditionally stable to the particular process and so will not limit the time step. The implicit method is simplest for linear terms and so we need to justify linearizing the free-surface terms. Free surface variations are typically much smaller than the nominal depth of the ocean

$$|\eta| << H$$

so that we are justified in linearized the free-surface equation:

$$\partial_t \eta + \nabla \cdot \int_{-H}^{0} \vec{v}_h \, dz = (P - E)$$

The integral for pressure can be cast into three contributions

$$p = \int_z^\eta g\rho \, dz = \int_0^\eta g\bar{\rho} \, dz + \int_0^\eta g(\rho - \bar{\rho}) \, dz + \int_z^0 g\rho \, dz$$

or

$$p \approx g\bar{\rho}\eta + g(\rho - \bar{\rho})\eta + \int_z^0 g\rho \, dz$$

To linearize the free-surface terms we assume that

$$(\rho|_{z=\eta} - \bar{\rho}) << \bar{\rho}$$

so that the pressure is approximated as

$$p \approx g\bar{\rho}\eta + \int_z^0 g\rho \, dz$$

We'll now discretize in time using the implicit backward method for the free surface terms:

$$\begin{aligned} \vec{v}_h^{n+1} + \Delta t g \nabla \eta^{n+1} &= \vec{v}_h^n + \Delta t \vec{G}_h \\ \eta^{n+1} + \Delta t \nabla \cdot \int_{-H}^{0} \vec{v}_h^{n+1} \, dz &= \eta^n + \Delta t (P - E) \end{aligned}$$

Eliminating $\vec{v}_h^{n+1}$ from the last equation by substitution from the first gives

$$\eta^{n+1} - \Delta t^2 \nabla \cdot gH \nabla \eta^{n+1} = \eta^n + \Delta t (P - E) - \Delta t \nabla \cdot \int_{-H}^{0} (\vec{v}_h^n + \Delta t \vec{G}_h) \, dz$$

This elliptic equation for η^{n+1} is solved at each step in the model and then the momentum equations can be stepped forward. Note that the equation is very similar to the elliptic equation for p_s in the pressure method and in fact can be recovered by allowing the time step to become very large.

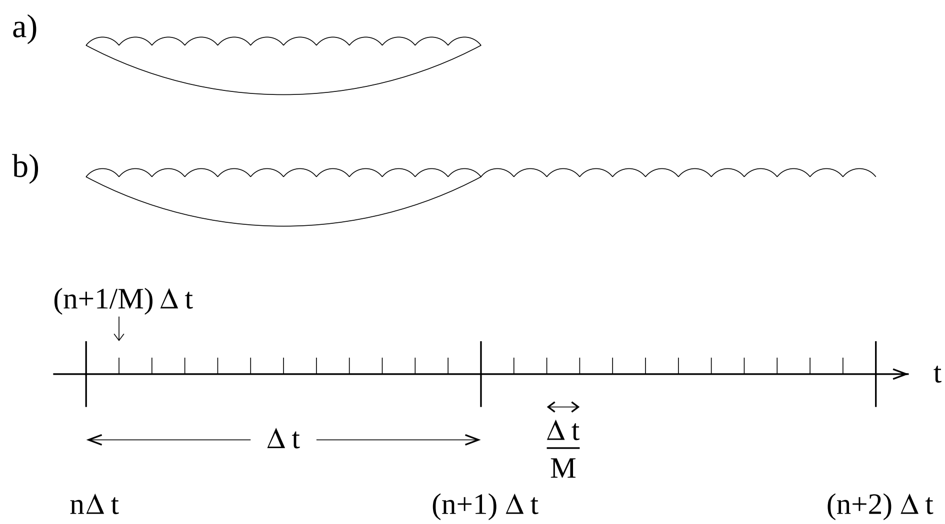

Figure 9.1: Two schemes for integrating the split barotropic and baroclinic modes. The long baroclinic time step, Δt, is divided into M smaller barotropic time steps. a) The barotropic mode is stepped forward M times and used at times $n\Delta t$ and $(n+1)\Delta t$, for example with trapezoidal weights. b) The barotropic mode is stepped forward $2M$ times and the average used at $(n+1)\Delta t$.

9.13 Split-explicit method

Since the limitation on time-step due to external gravity waves is due only to the depth integrated equations we can try split out that part of the system and integrate it with a shorter time step than the rest of the model.

We first obtain an approximation for the barotropic momentum equations

$$\partial_t < \vec{v}_h > + g\nabla\eta = < \vec{G}_h >$$

where $< \vec{G}_h >$ is the depth average of all the terms in the full momentum equations. We then integrate this equation with the free-surface equation forward using a short time step, $\Delta t/M$, where Δt is the regular time-step of the full model. One method for doing this is the forward-backward method which is as follows:

$$\eta^{n+(m+1)/M} = \eta^{n+m/M} + \frac{\Delta t}{M}\nabla \cdot (H + \eta^{n+m/M})\vec{v}_h^{n+m/M}$$

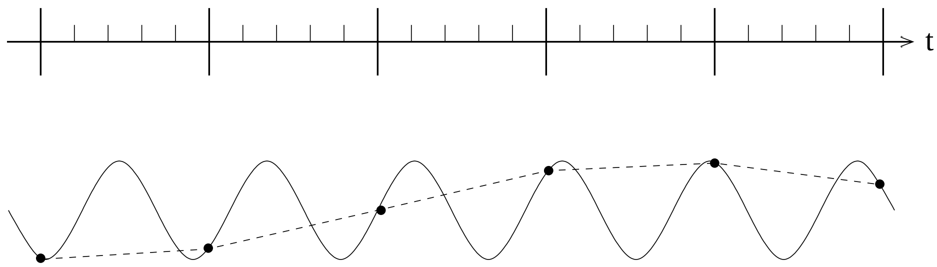

Figure 9.2: Higher frequency barotropic motions resolved with the short time step are aliased to lower frequencies if simply sub-sampled on to the baroclinic time line.

$$\bar{\vec{v}}_h^{n+(m+1)/M} \quad = \quad \bar{\vec{v}}_h^{n+m/M} + \frac{\Delta t}{M}(g\nabla\eta^{n+(m+1)/M} + < \vec{G}_h >$$

These equations can then be stepped forward M times to give an approximation for η^{n+1}. This method is schematically illustrated in Fig. 9.1a. However, simply using the value of η^{n+1} in the full forward equations essentially subsamples the high frequency barotropic motions and consequently alias high frequency energy onto lower frequencies, as depicted in Fig. 8.2. One solution is to average over the baroclinic time step and this is best done by integrated the barotropic equations forward over $2\Delta t$, as in Fig. 9.1b. Using the time average of η over the $2\Delta t$ interval filters out the high frequencies.

If we integrate the full equations forward using the long time step but using the time averaged η, we would arrive at two different estimates of the barotropic mode; one obtained from the full equations with long time step and one from the depth averaged equations with short time step. One remedy is to replace the barotropic part of the full solution with that of the resolved solution:

$$\vec{v}_h^{n+1} = \vec{v}_h^{n+1} - < \vec{v}_h^{n+1} > + \bar{\vec{v}}_h^{n+M/M}$$

There are many variants on this theme. For example, the original proposals used modal decompositions, thereby splitting the baroclinic and barotropic modes. The particular details of how the splitting is defined, the fast modes time averaged and the coupling implemented varies from model to model. These details make the method a little trickier than is first apparent but does have the advantage of being fully explicit (i.e. not requiring a matrix inversion associated with the elliptic equation). Another caveat is if the baroclinic time-step is long compare to other frequencies (such as Coriolis), then the barotropic component of those terms should also be included in the fast equations.

9.14 Accelerating the approach to equilibrium (Bryan 1984)

The turnover time of the deep ocean is of the order of 1000 years. This means that spinning up an ocean climate model to "equilibrium" requires at least a 1000 years of model time. We can estimate how long this would take using the following figures. Typical GCMs require approximately 200 floating point operations (simple computations) per grid point per step. A 4 degree model (90×45 grid points using geographic coordinates) with say 20 levels has 81000 points and with a 1 hour time-step would require of order 9×10^6 steps requiring a total of $7 \times 10^1 1$ operations. Modern PC processors can deliver around 500 MFlops (500×10^6 floating point operations per second) so to spin-up such a model would require over 3 days of uninterrupted computation. Bear in mind that in the early days of ocean modeling the computers were much smaller so this estimate leads to the order of weeks to months of computation. Even today, peak performance is unlikely, the number of computations per step is increased with more complicated parameterizations and the actual time-scale of adjustment might be many thousands of years. Such considerations push the turn around time for the calculation up to the orders of weeks.

Bryan, JPO 1984, analyzed a method of accelerating the convergence of ocean-climate models toward equilibrium. The approach manages to reduce the cost of computation by at least an order of magnitude and so is widely used even today. The consequences have been analyzed extensively, for example by Danabasoglu, McWilliams and Large, J. Clim 1996 who advocate using the Bryan acceleration method followed by a short period of synchronous integration before running climate sensitivity calculations.

The method is based on modifying the time-scales of the system so as to allow longer effective time steps but is equivalent to use different time steps for the thermodynamic tracer equations and the momentum equations. First, let derive the dispersion relation for internal inertia-gravity waves before looking at the distorted physics.

9.14.1 Vertical modes

Analysis of wave motions will make use of the linearized equations of motion

$$\begin{aligned}
\partial_t \vec{v}_h + f\hat{k} \wedge \vec{v}_h + \frac{1}{\bar{\rho}}\nabla_h p &= 0 \\
-b + \frac{1}{\bar{\rho}}\partial_z p &= 0 \\
\nabla_h \cdot \vec{v}_h + \partial_z w &= 0 \\
\partial_t b + wN^2 &= 0
\end{aligned}$$

where we are using a buoyancy variable, $b = -g\rho/\bar{\rho}$, with a large background stratification, $N^2 = \partial_z b_o(z)$. To avoid the complications of working with five equations in three dimensions it is convenient to describe the vertical structure in terms of modes. We'll illustrate this process first using simple Fourier modes which need us to assume that the background stratification is constant ($N^2 = const$). Each field can be described as a set of time and space dependent coefficients multiplying a vertical structure:

$$(\vec{v}_h, w, p, b) = \sum_m (\vec{v}_m, w_m, p_m, b_m)\, e^{imz}$$

so that vertical derivatives can be replaced, $\partial_z \to im$. Here $m = 2\pi/h_m$ is the vertical wave number. The last three equations become

$$\begin{aligned}
b_m &= imp_m \\
imw_m &= -\nabla_h \cdot \vec{v}_m \\
\partial_t b_m &= N^2 w_m
\end{aligned}$$

which can be combined into an equation for $p_m/\bar{\rho}$:

$$m^2 \partial_t \frac{p_m}{\bar{\rho}} + N^2 \nabla_h \cdot \vec{v}_m = 0$$

Thus, each vertical mode is governed by equations

$$\begin{aligned}
\partial_t \vec{v}_m + f\hat{k} \wedge \vec{v}_m + \nabla_h \frac{p_m}{\bar{\rho}} &= 0 \\
\partial_t \frac{p_m}{\bar{\rho}} + \frac{N^2}{m^2}\nabla_h \cdot \vec{v}_m &= 0
\end{aligned}$$

These take the form of the shallow water equations. Each vertical mode is governed by shallow water dynamics with a gravity wave speed, N/m. The dispersion relation for internal waves is now obtained by assuming a form $e^{i(kx+ly-\omega t)}$ for each vertical mode. This gives

$$\omega^2 = f^2 + \frac{N^2}{m^2}(k^2 + l^2)$$

Note that in Bryan, 1984, the modal decomposition is more general allowing for vertical variations in N^2. The resulting equation are expressed in terms of an equivalent depth H'_m associated with each mode.

9.14.2 Slowing down inertia-gravity waves

The basic method of acceleration is to use a shorter time step in the momentum equations, $\Delta t_{\vec{v}}$ than else where (namely the thermodynamic tracer equations, Δt):

$$\Delta t_{\vec{v}} = \frac{1}{\alpha}\Delta t$$

where α is a distortion factor. This essentially scales all terms in the momentum equation by α so that the equations become

$$\begin{aligned}
\partial_t \vec{v}_h + \frac{1}{\alpha}\left(f\hat{k} \wedge \vec{v}_h + \frac{1}{\bar{\rho}}\nabla_h p \right) &= 0 \\
-b + \frac{1}{\bar{\rho}}\partial_z p &= 0 \\
\nabla_h \cdot \vec{v}_h + \partial_z w &= 0 \\
\partial_t b + wN^2 &= 0
\end{aligned}$$

The subsequent analysis in Bryan, 1984, introduces a "stretched" time, $t' = t/\alpha$, in order to interpret the equations. Here, we will simply repeat the derivation of internal wave frequency above but including the distortion factor. The resulting equations for each vertical mode are

$$\begin{aligned}
\alpha \partial_t \vec{v}_m + f\hat{k} \wedge \vec{v}_m + \nabla_h \frac{p_m}{\bar{\rho}} &= 0 \\
\partial_t \frac{p_m}{\bar{\rho}} + \frac{N^2}{m^2}\nabla_h \cdot \vec{v}_m &= 0
\end{aligned}$$

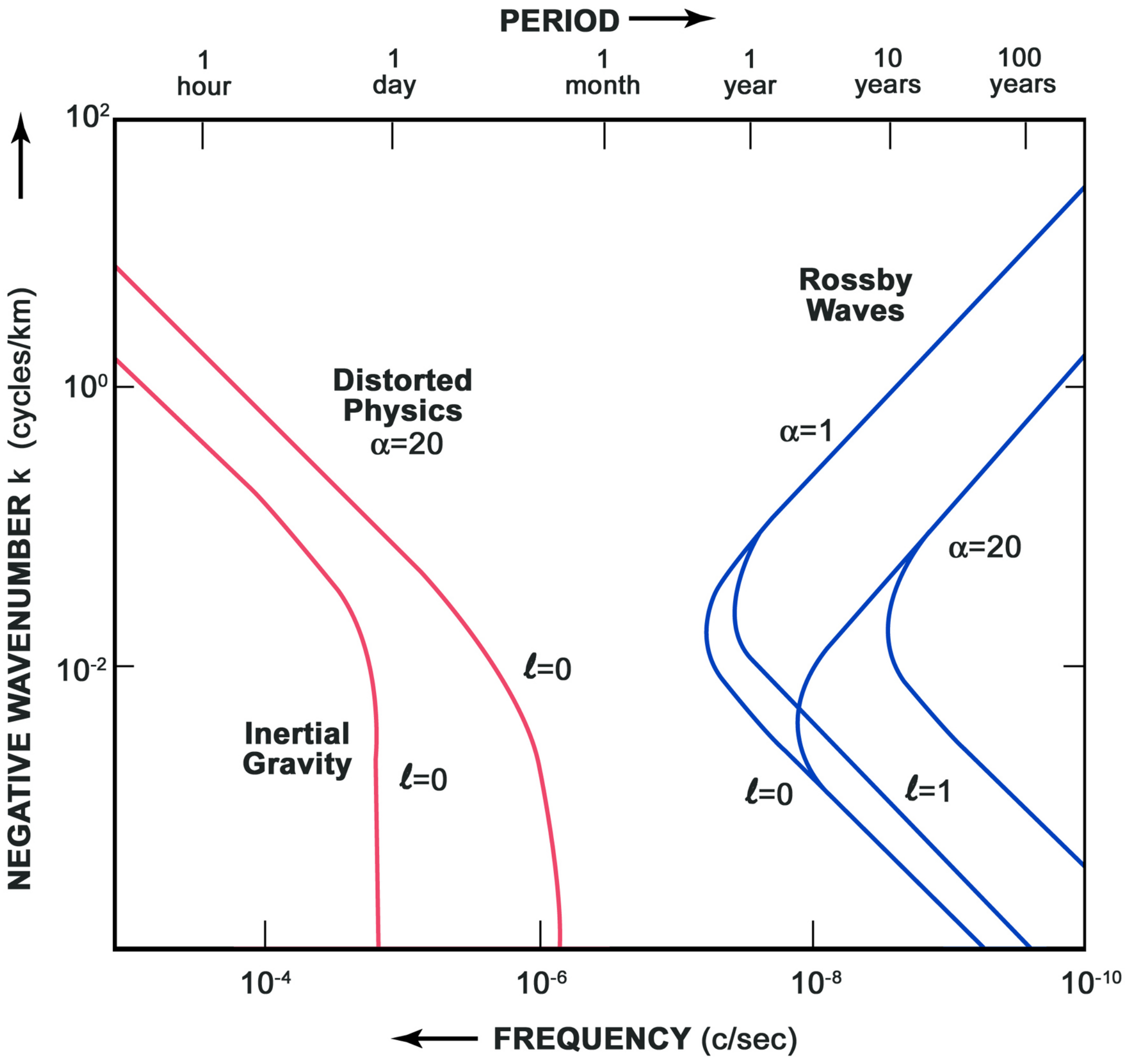

Figure 9.3: Dispersion diagram for mid-latitude waves corresponding to a radius of deformation of 50 km. The distortion factor α is equal to 20. Solid lines are for undistorted physics and dashed lines are for distorted physics. *Reproduced from Bryan, JPO 1984.*

so that the dispersion relation of the distorted internal waves is

$$\omega^2 = \frac{f^2}{\alpha^2} + \frac{N^2}{\alpha m^2}(k^2 + l^2)$$

Fig. 9.3 shows the undistorted and distorted frequencies using $\alpha = 20$ for pure zonally propagating inertia-gravity waves ($l = 0$). The distorted frequencies (dashed curve) are everywhere lower than the undistorted physics (solid curve). For long waves ($k^2 + l^2 << N^2/(f^2 m^2)$) where the frequencies should approach the Coriolis frequency, the distorted frequency is lower by α, while the gravity wave speed is $\sqrt{\alpha}$ slower.

Using a smaller time-step in the momentum equation than in the buoyancy equation slows the fast waves. Since time has been measured with respect to time in the buoyancy equation the allowed buoyancy time-step, Δt is larger than without the distortion.

9.14.3 Distorted Rossby waves

Using different time steps in the momentum and buoyancy equations affects all model time-scales. Rossby waves are also distorted. Geostrophic balance is unaffected by α since the Coriolis and pressure gradient terms are equally scaled by α:

$$f\hat{k} \wedge v_m = \frac{1}{\bar{\rho}} \nabla_h p_m$$

so that vorticity can be expressed as

$$\zeta_m = \frac{1}{f} \nabla_h^2 \frac{p_m}{\bar{\rho}}$$

The vorticity equation is

$$\alpha \partial_t \zeta_m + f \nabla_h \cdot \vec{v}_m + \beta v_m = 0$$

so that the potential vorticity equation is

$$\frac{\alpha}{f} \partial_t \nabla_h^2 p_m - f \frac{m^2}{N^2} \partial_t p_m + \frac{\beta}{f} \partial_x p_m = 0$$

The corresponding dispersion relation for distorted Rossby waves is

$$\omega = \frac{-\beta k}{\alpha(k^2 + l^2) + f^2 m^2 / N^2}$$

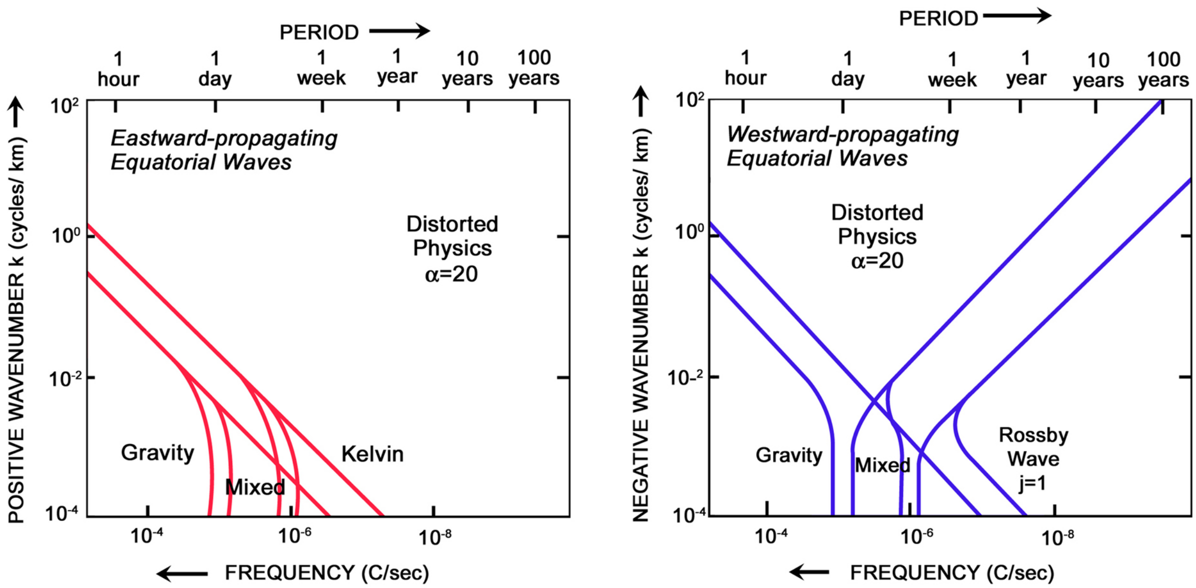

Figure 9.4: Dispersion diagram for equatorially trapped waves. The equatorial radius of deformation is 50 km and the distortion factor is 20. Left panel shows eastward propagating waves and right panel shows westward propagating waves. Solid lines are for undistorted physics and dashed lines are for distorted physics. *Reproduced from Bryan, JPO 1984.*

and is plotted in Fig. 9.3. For long waves ($k^2 + l^2 << 1/L_r^2$) then the distortion becomes negligible. The maximum frequency is reduced when $\alpha > 1$. Because short waves are slowed down the distortion can affect the stability of models at western boundaries; the width of a western boundary layer can be thought of as the distance a short and slow eastward propagating wave can travel before being dissipated.

Bryan, 1984, goes on to analyze the distortion of equatorial waves, the pertinent results are reproduced in Fig. 9.4. The linear shallow water equations on a β-plane

$$
\begin{aligned}
\alpha \partial_t u_m - \beta y v_m + \partial_x \frac{p_m}{\bar{\rho}} &= 0 \\
\alpha \partial_t v_m + \beta y u_m + \partial_y \frac{p_m}{\bar{\rho}} &= 0 \\
\partial_t \frac{p_m}{\bar{\rho}} + \frac{N^2}{m^2} \nabla \cdot \vec{v}_m &= 0
\end{aligned}
$$

can be reduced to a single equation of the form

$$\partial_t \left[\alpha^2 \partial_{tt} + (\beta y)^2 - \alpha \frac{N^2}{m^2} \nabla_h^2 \right] v_m - \beta \frac{N^2}{m^2} \partial_x v_m = 0$$

The usual procedure of rescaling the meridional coordinate to

$$y' = \beta^{1/2} y \left(\frac{\alpha m^2}{N^2} \right)^{1/4}$$

shows that the scale of equatorially trapped waves is weakly modified. The Kelvin wave speed

$$c = \left(\frac{N^2}{\alpha m^2} \right)^{1/2}$$

is reduced by $\sqrt{\alpha}$.

Baroclinic instability is also distorted leading to lower growth rates but also a lower threshold for instability. We will not repeat his analysis here but the essential results are the same as for the mid-latitude waves; all physical time scales are slowed as a result of the acceleration technique.

The method outlined above reduces the computational cost of reaching equilibrium by an order of magnitude or two, depending on the particular configuration. However, he outlined a further acceleration where the buoyancy equation is integrated with longer time steps at depth than at the surface. We can write the distorted linear buoyancy equation as

$$\partial_t b + \frac{w N^2}{\gamma(z)} = 0$$

where $\gamma(z)$ is 1 at the surface and smaller at depth. Now, since $N^2/\gamma(z)$ is clearly a function of depth the modal decomposition must reflect the appropriate vertical structure and the Fourier decomposition we used above will not work. The more general approach is used in Bryan, 1984. Rather than reproduce the analysis here we simply make some observations about the method. The approach allows for a further order of magnitude increase in time step from top to bottom. However, the vertical structure of the modes must be affected by the choice of $\gamma(z)$ since $N^2/\gamma(z)$ can be interpreted as a modified stratification $N'^2(z)$. This begs the question of how an accelerated model can achieve the correct vertical structure. Danabasoglu et al., 1995, found that once the accelerated model is equilibrated, further adjustment without acceleration is necessary to obtain the true equilibrium. There is also a problem with conservation using deep acceleration with diffusion and/or a flux divergence form of advection. Because the flux is multiplied by a different time step at different levels the sum of tendencies, in which the fluxes should cancel is not zero. Heat and salt can therefore be created/lost when using deep acceleration. As a result, deep acceleration is not as widely used though does appear to help speed up the convergence to equilibrium.

One final comment about the general acceleration approach is that convergence to an equilibrium is most meaningful if there is a steady state solution (i.e. one forced by steady forcing). Recent examination of seasonally forced models indicates that the seasonal cycle can be distorted (for large α) to the point that the climate (quasi-equilibrium state) is affected.