7. WKB Theory For internal Gravity Waves – Non-rotating Case

Slowly Varying Medium

So far we have considered N = constant, Now we shall assume that N is a slowly varying function with respect to the phase of the wave (N represents the medium).

N is a much stronger function of z than of (x,y) so let us consider N = N(z) only. Then from the ray equations, that describe the motion of a wave package in a slowly varying medium, we have

$$\frac{\partial \overline{K}}{\partial t} + \vec{c}_g \bullet \nabla \vec{K} = -\nabla \Omega \qquad \text{where } \Omega = N\frac{K_H}{K}$$

that is

$$\frac{\partial k}{\partial t} + \vec{c}_g \bullet \underline{\nabla} k = 0 \Rightarrow k = k_o$$ initial value is conserved!

$$\frac{\partial l}{\partial t} + \vec{c}_g \bullet \underline{\nabla} l = 0 \Rightarrow l = l_o$$ initial value is conserved

$$\frac{\partial m}{\partial t} + \vec{c}_g \bullet \underline{\nabla} m = -\frac{\partial \Omega}{\partial z}$$ only m is changed when the wave goes through a region of variable N

Also

$$\frac{\partial \omega}{\partial t} + \vec{c}_g \bullet \underline{\nabla} \omega = \frac{\partial \Omega}{\partial t} = 0$$ $\omega = \omega_o$ initial value

(k,l,ω) are constant for an observer moving with the wave packet (but vary at a fixed position). The governing equation is the same:

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2(z)\nabla_H^2 w = 0 \qquad (1)$$

Look for a solution of the form:

$$W = A(z)e^{i(kx+ly-\omega t+\theta(z))}$$

$$\text{with} \quad \frac{d\theta}{dz} >> \frac{dA}{dz} \quad \text{and} \quad \frac{d^2A}{dz^2} << A \quad \begin{cases} \frac{d}{dz} \sim o(\frac{1}{\lambda}) \\ \frac{dA}{dz} \sim o(\frac{1}{L_m}) \\ \frac{d^2A}{dz^2} =\sim \theta(\frac{1}{{L_m}^2}) \end{cases}$$

As N is varying slowly, locally the solution will look like a plane wave. Then we define

$$m(z) = \frac{d\theta}{dz}$$

(For rigorously plane wave in a homogeneous medium θ = mz and $m = \frac{d\theta}{dz} = \text{constant}$)

Inserting the solution into the governing equation (1) we have

$$-\omega^2[-(k^2 + l^2)A + A_{zz} - A\theta_z^2] - N^2(k^2 + l^2)A \quad \text{real part}$$

$$-i\omega^2(2\theta_z A_z + \theta_{zz}A) = 0 \quad \text{imaginary part}$$

or

$$A_{zz} + A[\frac{(N^2 - \omega^2)K_H^2}{\omega^2} - \theta_z^2] - 2i\omega^2\theta_z^{1/2}\frac{\partial}{\partial z}(A\theta_z^{1/2}) = 0$$

Real and imaginary parts must both be zero. We assumed $\theta_z \underset{\sim}{}$0(1) and $A_{zz}/A<<1$, then the dominant term in the real part is

$$\frac{(N^2 - \omega^2)}{\omega^2}K_H^2 - \theta_z^2 \simeq 0 \quad \text{which gives}$$

$$m^2 = \theta_z^2 = \frac{N^2 - \omega^2}{\omega^2}K_H^2$$

$$m(z) = \frac{\partial\theta}{\partial z} = (\frac{N^2 - \omega^2}{\omega^2})^{1/2}K_H$$

$$w = A(z)e^{i(kx + ly - \omega t + \theta(z))} \quad \text{and}$$

$$-\omega^2[-(k^2+l^2)A+\frac{\partial^2}{\partial z^2}w]-N^2(k^2+l^2)A=0$$

$\frac{\partial}{\partial z}w=\frac{dA}{dz}e^{i(...)}+iA\frac{d\theta}{dz}e^{i(...)}$ and the second differentiation gives

$$\frac{\partial^2 w}{\partial z^2}=\frac{d^2A}{dz^2}e^{i(kx+ly-\omega t+\theta(z))}+i\frac{dA}{dz}\frac{d\theta}{dz}e^{i(...)}+iA\frac{d^2\theta}{dz^2}e^{i(...)}-A(\frac{d\theta}{dz)})^2e^{i(...)}$$

or

$$-\omega^2[-(k^2+l^2)A+A_{zz}-A\theta_z^2]-\omega^2(k^2+l^2)A-i\omega^2[2A_z\theta_z+A\theta_{zz}]=0$$

Then the vertical phase factor is

$$\theta(z)=\int_{z_o}^{z}K_H\sqrt{\frac{N^2(z)-\omega^2}{\omega^2}}dz \qquad \text{as N(z) is varying}$$

z_o = initial position

We must equate to zero also the imaginary part of the equation for A:

$\frac{\partial}{\partial z}(A\theta_z^{1/2})=\frac{\partial}{\partial z}(Am^{1/2})=0$ => $Am^{1/2}$ = constant = $A(z_o)m_o^{1/2}$

or

$A(z)=\frac{A(z_o)}{(m/m_o)^{1/2}}$ As m gets larger, i.e. in a region of larger N, A decreases

If the wave propagates to a z_T where $\omega>N$, m becomes im, the solution becomes exponentially decaying beyond z_T. In other words, N(z) acts as an index of refraction for w, and $\omega=N$ is the point of total reflection. Then we can find the point of total reflection, or the turning point for the wave packet (the ray) by using the ray equations. For two dimensions:

$$\frac{dz}{dt} = c_{gz} = -N\frac{km}{K^3}$$

$$\frac{dx}{dt} = c_{gx} = +N\frac{m^2}{K^3}$$

$$\frac{dz}{dx} = \frac{c_{gz}}{c_{gx}} = -\frac{k}{m}$$

or

the ray path

Rewriting $m = (\frac{N^2 - \omega^2}{\omega^2})^{1/2} k$ as $\omega^2 = \frac{N^2(z)k^2}{m^2 + k^2}$

$$\frac{k}{m} = \frac{\omega}{\sqrt{N^2 - \omega^2}} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\omega}{\sqrt{N^2 - \omega^2}}$$

Consider the region near z_T where $N(z_T) \simeq \omega$. Expand N^2 around z_T, the turning point

$$N^2(z) = N^2(z_T) - (z_T - z)\frac{dN^2}{dz}\Big|_{z_T} + \dots$$

So $(N^2 - \omega^2)_{z_T} = -(z_T - z)\frac{dN^2}{dz}\Big|_{z_T}$

and $\dfrac{dz}{dx} = \dfrac{-\omega}{\sqrt{(-\frac{dN^2}{dz}\big|_{z_T})(z_T - z)}}$

or $(z_T - z)^{1/2} dz = \dfrac{-\omega \ dx}{\sqrt{-(\frac{dN^2}{dz})_{z_T}}}$

But

$$\int_{z_T}^{z} (z_T - z)^{1/2} = \frac{2}{3}(z_T - z)^{3/2} = -\frac{\omega}{\sqrt{-(\frac{dN^2}{dz})_{z_T}}} \int_{x_T}^{x} dx = \frac{+\omega}{\sqrt{-(\frac{dN^2}{dz})_{z_T}}}(x_T - x)$$

and

$$z_T - z = [\frac{3}{2}\frac{\omega}{\sqrt{-(\frac{dN^2}{dz})_{z_T}}}]^{2/3}(x_T - x)^{2/3}$$

The ray path has a cusp at the turning point (x_T, z_T).