

Lecture 6

6.1 Administration

- Collect PS3
- Distribute PS4 (due next Thursday 2nd)
- Distribute PS2 and quiz
- **QUIZ!**
- **From homework:** $\nabla \cdot \omega = 0$ for any flow?

For this to hold, one must assume $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$. When is this valid? When \mathbf{u} is continuous and differentiable. In a shock, this does not hold. This might also not be satisfied in a numerical calculation.

6.2 Further discussion of the linear uncertainty propagator

Recall (noting figure 6.1):

- $\text{SVD}(\mathbf{M}) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$
- $\mathbf{U} = \text{eig}(\mathbf{M}\mathbf{M}^T) \Rightarrow$ directions of principle axes of final time ellipse.
- $\mathbf{V} = \text{eig}(\mathbf{M}^T\mathbf{M}) \Rightarrow$ directions of initial-time circle that evolves into the axes of the final-time ellipse.
- $\mathbf{\Sigma} = \sqrt{\text{eigval}(\mathbf{M}\mathbf{M}^T)} = \sqrt{\text{eigval}(\mathbf{M}^T\mathbf{M})} \Rightarrow$ factors by which the vectors grow.

This is really neat! We can be sitting at a time 0 and figure out what directions at time 0 are going to experience the most growth over Δt , and know by what factor they will grow! (See figure 6.2).

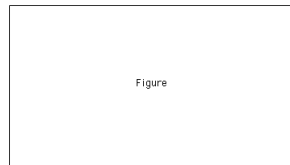


Figure 6.1: (`fig:Lec6LinUncerProp`) The linear uncertainty propagator, \mathbf{M} , acting on an initial state.

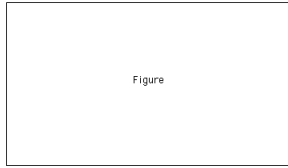


Figure 6.2: (`fig:Lec6LinUncerPropFull`) Again, the linear uncertainty propagator, \mathbf{M} , acting on an initial state. Here the initial vectors (v_1 and v_2) and final vectors ($\sigma_1 u_1$ and $\sigma_2 u_2$) are noted. $\mathbf{M}v_i = \sigma_i u_i$.

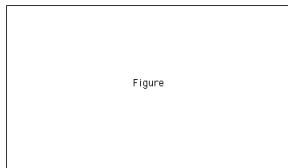


Figure 6.3: (`fig:Lec6InitialEllipse`) With an initial ellipse (not sphere), you do not find the directions that you want.

Note: SVD assumes you have a sphere at initial time. If you don't have a sphere, the SVD will not find the directions you want (see figure 6.3). If this is the case, you need to apply a norm to make initial ellipse into a sphere (**Show rock**).

Who cares about this stuff? Lots of people do. So far we've been concentrating on the situation where we have a blob of fluid surrounded by other fluid. There is a kinematically identical situation where instead of focusing on a region of the fluid, we consider the region of the fluid, we consider the region that the entire fluid is occupying as a whole: the fluid's state space.

State space... what's that? It is sometimes called phase space. It is the space that contains the trajectories of the system as a whole. The state space has the dimension of the number of degrees of freedom of your system. At each instant in time an atmosphere model with 10^{16} degrees of freedom occupies a single point in the state space.

The Lorenz attractor is a good example of points in state space (see figure 6.4). The system has three degrees of freedom, and at each instant in time the state can be represented as a point in this space. One need not limit themselves to 3D – Could also be in 10^6 D.

When I take a blob in state space, each point in that blob is a completely different system state. Just like the case where we have a fluid blob, this state space blob will translate, rotate and deform. Why should we care? If you want to make a forecast you need an initial condition, and we all know that small errors in IC's can translate to large forecast errors. Lets say we have errors in our initial conditions, and the ball in state space represents our uncertainty associated with the first guess. We can get uncertainty in our forecast by propagating the entire ball forward and saying the forecast must lie in the ellipse. This could be expensive. Instead we might just want to know the

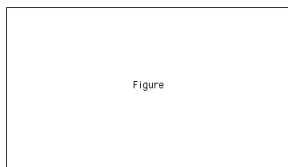


Figure 6.4: (`fig:Lec6LorenzAttractor`) The Lorenz attractor.

axes of the ellipse... the singular vectors for that location in state space and for that forecast period.

Examples: Seen figures 6.5 through 6.7. The figure captions contain the details.

6.3 Reynold's transport theorem

We learned about the material derivative and how it relates Lagrangian rates of change to Eulerian rates of change. (**QUIZ 1**).

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (6.1)$$

(**QUIZ 2**). Turns out the material derivative can be used for intensive properties, but not for extensive properties.

Intensive: Independent of the size of the system. Not additive. Examples: temperature, pressure, velocity, per volume, per mass.

Extensive: Depends on the size or extent of the system. Is the sum of its parts. Examples: mass, total energy, momentum. Note extensive properties tend to be subject to conservation or balance laws.

I alluded in an earlier lecture that the conservation laws are “easy”, or at least familiar, in a Lagrangian framework. It is the Reynold's Transport Theorem that take us from Lagrangian to Eulerian when we are dealing with extensive properties.

Consider some extensive property, C . C is some stuff that is the sum of its parts (mass, concentration, energy). Now define c , which is stuff per volume:

$$C = \iiint_{\mathcal{V}} c \, d\mathcal{V} \quad (6.2)$$

Here \mathcal{V} denotes a Lagrangian volume. Now, let's pretend we want to know how C changes with time, that is:

$$\frac{dC}{dt} = \frac{d}{dt} \iiint_{\mathcal{V}} c \, d\mathcal{V} \quad (6.3)$$

If we were staying in the Lagrangian framework, we'd be done. But we aren't, so we're not. Consider an Eulerian box as seen in figure 6.8. At time t , let the Lagrangian volume \mathcal{V} coincide with the Eulerian volume V . In this case, since I am fixed to a control volume (doesn't vary as a function of time), the time derivative on the RHS in equation 6.3 can be taken inside:

$$\frac{dC}{dt} = \iiint_{\mathcal{V}} \frac{\partial c}{\partial t} \, d\mathcal{V} \quad (6.4)$$

But this only gives the local change in C . We are now savvy students of fluid dynamics and know that we have to take into account the motion of the fluid through the control volume. Note from figure 6.8 that there will be a contribution to $\frac{dC}{dt}$ from stuff entering V (area A) and from stuff leaving V (area B). This needs to be added to our expression of $\frac{dC}{dt}$. Need to add the contribution of B and subtract the contribution of A .

Recall, $C = c \times \text{volume}$, so we need to know the volume of regions A and B . Look at a segment of area B as seen in figure 6.9. The change in C due to this little volume of the outflow is

$$\Delta C = c \Delta x \Delta A \quad (6.5)$$

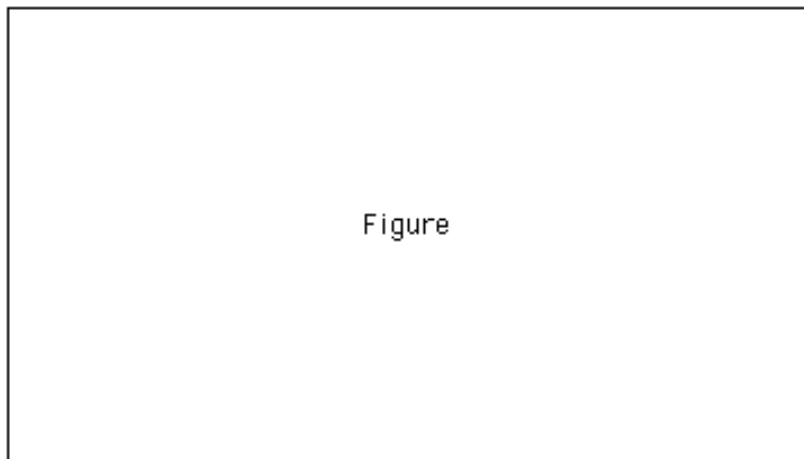
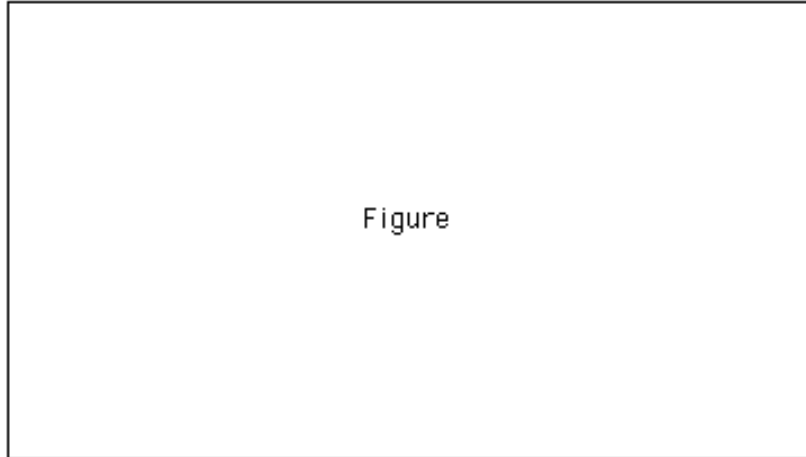


Figure 6.5: (fig:Lec6Ileda1 through fig:Lec6Ileda1) Here is an example of a chaotic system as the ball goes to the ellipse. I can find the major axes of the ellipse using SVD on the linear propagator... this gives me a worst case scenario for any forecasts (assuming (**Jim - Can't read your notes here, lecture 6 page 9**) holds). **Jim - There is also a couple post-it notes on the figures in the note for which your writing did not come off to clearly, so I was not able to add those comments, same with next two examples as well, Atm SV and ENSO.**

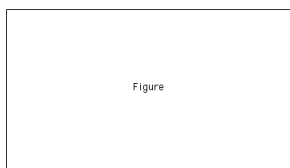


Figure 6.6: (`fig:Lec6AtmSV`) Streamfunction at 850 hPa, 3 day optimization time. This type of analysis happens all the time in NWP (**Jim – What does this stand for?**) and atmospheric science (ensemble forecasting, sensitivity studies).

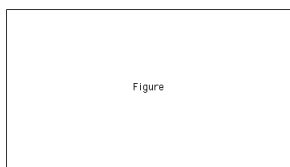


Figure 6.7: (`fig:Lec6ENSO`) Oceanography examples. ENSO forecasting: what are things sensitive too? The above is a plot of SST; initial and final-time. Climate: what is overturning sensitive too?

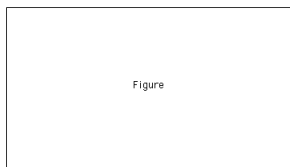


Figure 6.8: (`fig:Lec6EulerianBox`) Eulerian box. used for comparing Eulerian volume, V and Lagrangian volume \mathcal{V} . The volumes coincide at time t .

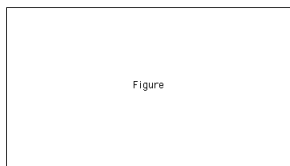


Figure 6.9: (`fig:Lec6AreaB`) Blow up of area B from eulerian box seen in figure 6.8.

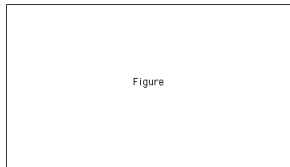


Figure 6.10: (fig:Lec6DivThrmSurf) Surface, A , where interior is chopped up into a bunch of little volumes.

Δx perpendicular to ΔA will be determined by the component of velocity perpendicular to ΔA

$$\Delta x = \mathbf{u} \cdot \mathbf{n} \Delta t \quad (6.6)$$

$$\Delta C = c \mathbf{u} \cdot \mathbf{n} \Delta t \Delta A \quad (6.7)$$

$$\frac{\Delta C}{\Delta t} = c \mathbf{u} \cdot \mathbf{n} \Delta A \quad (6.8)$$

$$\frac{dC}{dt} = c \mathbf{u} \cdot \mathbf{n} \Delta A \quad (6.9)$$

$$= c \mathbf{u} \cdot d\mathbf{A} \quad (6.10)$$

Note that the sign is taken care of since \mathbf{n} is positive out. In B , $\mathbf{u} \cdot \mathbf{n} > 0$, and in A , $\mathbf{u} \cdot \mathbf{n} < 0$. Along top and bottom, $\mathbf{u} \cdot \mathbf{n} = 0$. Now integrate over area:

$$\iint_A c \mathbf{u} \cdot d\mathbf{A} \quad (6.11)$$

This is the component that we seek which must be added. Hence,

$$\frac{dC}{dt} = \iiint_V \frac{\partial c}{\partial t} dV + \iint_A c \mathbf{u} \cdot d\mathbf{A} \quad (6.12)$$

This is the Reynold's Transport Theorem. It states that the rate of change of C has a local component, $\frac{\partial}{\partial t}$, plus the impact of stuff entering and exiting the control volume, \iint_A . Note how groovy this is. We've take a Lagrangian expression of the rate of change of stuff

$$\frac{dC}{dt} = \frac{d}{dt} \iiint_{\mathcal{V}} c d\mathcal{V} \quad (6.13)$$

and turned is into the sum of an Eulerian volume integral and an Eulerian surface integral.

We can put the RTT into an alternative form by using the divergence, or Gauss' theorem (**QUIZ 3**). Let's take a quick aside and talk about the divergence theorem for a moment (a favorite from 2nd problem set). It states

$$\iiint_V \nabla \cdot \mathbf{Q} dV = \iint_A \mathbf{Q} \cdot d\mathbf{A} \quad (6.14)$$

$\iint_A \mathbf{Q} \cdot d\mathbf{A}$ is an expression for the flux, or flow, of \mathbf{Q} through the surface A . $\nabla \cdot \mathbf{Q} = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint \mathbf{Q} \cdot d\mathbf{A}$, that is, the divergence is the flux per unit volume.

Imagine some surface A , and chop the interior of the surface into a bunch of little volumes (see figure 6.10). It should be clear that the flux through A will be the same as the sum of the flux through all the little volumes:

$$\iint_A \mathbf{Q} \cdot d\mathbf{A} = \sum_{i=1}^n \iint_{A_i} \mathbf{Q} \cdot d\mathbf{A} \quad (6.15)$$

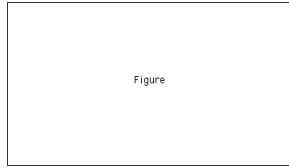


Figure 6.11: (`fig:Lec6DivThrmLittleVolume`) Little volumes, V_1 and V_2 . Fluxes on interior edges cancel.

Where A_i is the surface enclosing the little volume V_i .

To convince you of this, zoom in on a couple of these little volumes (see figure 6.11). The flux from V_1 into V_2 will be the same as the flux from V_2 into V_1 , but with the opposite sign. Their total contribution to the flux through A will be zero... they cancel. Of course, all subvolumes have common sides, except those subvolumes that border A .

Now for a little trick. Multiply and divide RHS by V_i :

$$\iint_A \mathbf{Q} \cdot d\mathbf{A} = \sum_{i=1}^n \left[\frac{1}{V_i} \mathbf{Q} \cdot d\mathbf{A} \right] \quad (6.16)$$

(**Jim – To the right of the above equation you have written “ dV ?”**). Now we let the subvolumes shrink to zero (let $n \rightarrow \infty$). By definition, the value in the brackets becomes the divergence of \mathbf{Q} . Divergence is the flux per unit volume. The $\iint \mathbf{Q} \cdot d\mathbf{A}$ is the flux and the $\frac{1}{V_i}$ is the per volume:

$$\iint_A \mathbf{Q} \cdot d\mathbf{A} = \lim_{n \rightarrow \infty, V_i \rightarrow 0} \sum_{i=1}^n \iint_{A_i} \mathbf{Q} \cdot dV_i \quad (6.17)$$

But in this limit, this sum is a volume integral, and the divergence theorem results:

$$\iint_A \mathbf{Q} \cdot d\mathbf{A} = \iiint_V \nabla \cdot \mathbf{Q} dV \quad (6.18)$$

Back to the RTT. Using the divergence theorem:

$$\frac{dC}{dt} = \iiint_V \frac{\partial c}{\partial t} dV + \iint_A c \mathbf{u} \cdot d\mathbf{A} \quad (6.19)$$

$$= \iiint_V \frac{\partial c}{\partial t} dV + \iiint_V \nabla \cdot (c \mathbf{u}) dV \quad (6.20)$$

$$= \iiint_V \left[\frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{u}) \right] dV \quad (6.21)$$

This is another form of RTT. Now, you all recall you vector calculus: $\nabla \cdot (a\mathbf{b}) = (\mathbf{b} \cdot \nabla)a + a(\nabla \cdot \mathbf{b})$:

$$\frac{dC}{dt} = \iiint_V \left[\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c + c \nabla \cdot \mathbf{u} \right] dV \quad (6.22)$$

$$= \iiint_V \left[\frac{Dc}{Dt} + c \nabla \cdot \mathbf{u} \right] dV \quad (6.23)$$

$$(6.24)$$

Here's a form of the RTT that states that rate of change of C with a material derivative part (which we should be familiar with), but also this divergence term. C can change with time locally, $\frac{\partial}{\partial t}$, through advection of c gradients $\mathbf{u} \cdot \nabla$, and through c getting smashed (or anti-smashed) into a point. We will get back to this divergence term in a bit.

6.4 Conservation of mass

With the Reynold's transport theorem, expressing the conservation of mass is trivial. The extensive property, $C = M$ is mass. The intensive version is density, mass per volume, $c = \rho$. We know from solid mechanics that

$$\frac{dM}{dt} = 0 \quad (6.25)$$

So,

$$\frac{dM}{dt} = 0 = \iiint_V \left[\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} \right] dV \quad (6.26)$$

This relationship holds for any volume, so for $\iiint = 0$ (there won't be + volumes and - volumes), we arrive at the continuity relationship (**QUIZ 4**):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \left[= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \quad (6.27)$$

6.5 Reading for class 7

KC01: 4.5 - 4.11