

# Lecture 13

## 13.1 Administration

- Collect PS 6
- Distribute PS 7 (due November 4th)
- There are many problems with this lecture:
  1. Crap job with Helmholtz number 1
  2. Assumptions that  $\frac{\rho}{p}$  is spatially constant in baroclinic situations
  3. Crap  $\frac{dz}{dt}$  stuff for d'Lamber's paradox

## 13.2 Last problem set

- Many shared code. This is perfectly fine but do your own write-up and comments.
- Many had trouble with my `step_it` routine (RK4) and wrote their own integration routine. First, I'm pleased you made the effort but unless you were very careful, this got you into trouble with part (b). The correct answer was a circle, but most who did their own integration got a spiral (see figure 13.1).

Why the problem? First consider and Euler step at is seen in figure 13.2. Here, updating the position consists of a single step:  $\Delta y = v_y \Delta t$ . If  $\Delta t$  is too big, then you'll end up with a state well off the correct trajectory. The error (derived through a Taylor expansion) is  $0(\Delta t^2)$ .

We can do a bit better using the midpoint method (also called a 2nd order Runge-Kutta) as is seen in figure 13.3. From your starting point you step forward  $\frac{1}{2}\Delta t$  using an Euler step. Then, calculate the velocity at the halfway point and apply *that* velocity over the entire interval. The error here is  $0(\Delta t^3)$ .

Better still is the 4th order Runge Kutta scheme as is seen in figure 13.4. The method for this goes as follows:

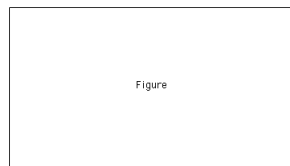


Figure 13.1: (`fig:Lec13Spiral`) The correct answer was a circle, but most who did their own integration got a spiral.

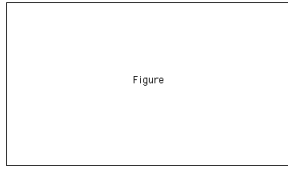


Figure 13.2: (`fig:Lec13EulerStep`) Euler method for stepping forward a state.

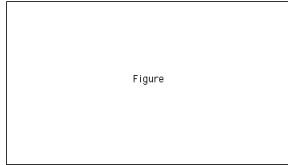


Figure 13.3: (`fig:Lec13MidpointStep`) Midpoint method for stepping forward a state.

1. Find the velocity at starting point.
2. Step forward  $\frac{\Delta t}{2}$  and calculate the velocity there.
3. Apply that velocity back at the starting point, but, unlike the midpoint method, step forward only  $\frac{\Delta t}{2}$  and calculate the velocity there.
4. Apply that velocity back at the starting point and step a full  $\Delta t$  forward. But this isn't the endpoint. Calculate the velocity there.
5. The final endpoint comes from combining all the increments produced:

$$\Delta y = \frac{v(\text{Step 1})}{6} + \frac{v(\text{Step 2})}{3} + \frac{v(\text{Step 3})}{3} + \frac{v(\text{Step 4})}{6} \quad (13.1)$$

### 13.3 Unknown

**Jim – Here was a note that you wrote that I was not quite sure where you wanted. Is this separate, part of the problem set, administration, or a lead in for the next section?**

Kundu's bathtub vortex example simply says that we wrote down the wrong form of the momentum equations when we proved Kelvin's theorem. We'll sort that out once we introduce the earth's rotation (it's coming soon!).

### 13.4 Helmholtz Vortex Theorems

Another important theorem, set of theorems actually, for vortex flows are the Helmholtz vortex theorems. Starting with the following assumptions (true for the Euler or perfect fluid):

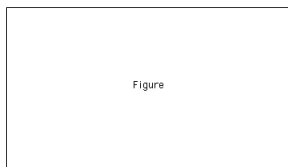


Figure 13.4: (`fig:Lec13RK4Step`) 4th order Runge-Kutta method for stepping forward a state.

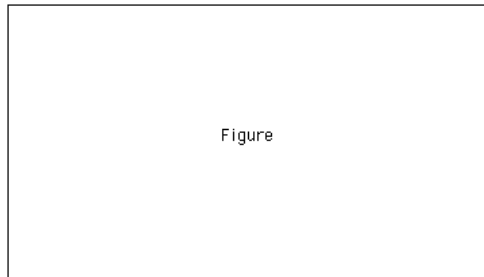


Figure 13.5: (fig:Lec13PassiveActive) In the left panel, a passive tracer being advected by an active one. In the right panel, two active tracers advecting each other.

1. inviscid
2. barotropic
3. conservative body forces
4. non-rotating force (proper derivation)

Helmholtz vortex theorems state that (**Q3**):

1. Vortex lines move with the fluid.
2. The circulation of a vortex tube is constant along its length.
3. A vortex tube can only end at a solid boundary or form a closed loop.
4. The circulation of a vortex tube is constant in time.

All of these follow from Kelvin’s theorem. The first theorem is the most far-reaching. It says that vortex lines are “pinned” to the flow. Think about that; vorticity is a dynamically active variable, yet it is moved by the flow as though it were a passive tracer. This is consistent with our point vortex model. A single point vortex does not self advect, it is moved by the velocity field generated by other point vortices. It is not entirely accurate to call a point vortex a passive tracer. A given vortex is influenced by the other vortices in the flow, but it influences them as well (see figure 13.5).

## 13.5 Proofs of Helmholtz Vortex Theorems

1. The proof Kundu gives for vortex lines moving with the fluid goes like this (see figure 13.6):

At  $t$ ,  $S$  lies on the surface of a vortex tube. Because lines of constant vorticity lie within  $S$  by construction, there is no component of vorticity normal to  $S$ , so the circulation of the curve bounding  $S$  is zero.

Kelvin’s theorem says  $\frac{d\Gamma}{dt} = 0$  so  $\Gamma$  at  $t + \Delta t$  also must be 0. For  $\Gamma = 0$ , this new surface at time  $t + \Delta t$  must also have no projection of vorticity normal to  $S'$  so lines of constant vorticity lie within  $S'$ , so  $S'$  also lies on the surface of a vortex tube. This must hold for all  $\Delta t$  so there is no chance that  $S'$  is on a different vortex tube. This must hold for all  $\Delta t$ , so there is no chance that  $S'$  is on a different vortex tube. (Note that this proof really says that  $S$  can’t move in such a ways as to generate vorticity in  $S$  (**Jim – The rest of this note got cut off.**)). Vortex tubes move. . . (**Jim – The rest of this got cut off too.**)

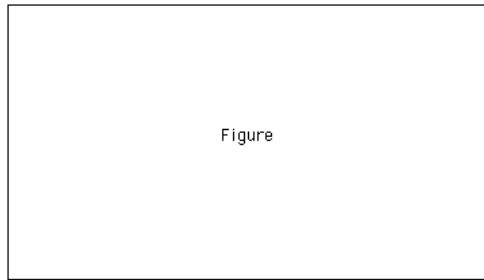


Figure 13.6: (fig:Lec13VortexLines) Vortex lines at  $t$  (left panel) and  $t + \Delta t$  (right panel).

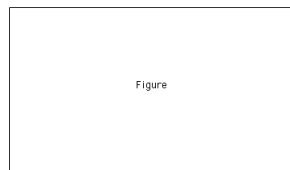


Figure 13.7: (fig:Lec13VortexTube1) A vortex tube with a surface around it that is everywhere parallel to the tube.

2. Helmholtz's 2nd theorem was that  $\Gamma$  is constant along the length of a vortex tube. Consider a vortex tube. Consider a vortex tube as is seen in figure 13.7. Put a surface around it that is everywhere parallel to the tube. There is no projection of vorticity in directions normal to the surface, so through Stokes' Theorem we know that the line integral of the velocity along the route  $abcd$  is zero.

Imagine the slit is infinitesimally small, then the contribution to the line integral of  $ab$  cancels the contribution of  $cd$  (equal and opposite). This means the contribution to the line integral of  $bc$  must cancel with  $da$ . This means that the circulations of the areas enclosed by the top and bottom of the cylinder must be the same (the "opposite" bit comes about from the opposite directions of the line integral at the top and bottom of the cylinder). So, the circulation along the length of the vortex tube is constant.

3. The 3rd theorem states that a vortex tube can't end within the fluid. We can use the same picture as before, but have the vortex tube end in the middle of the surrounding surface (see figure 13.8).

You still have the situation where there is no component of vorticity normal to the surface so by Stokes theorem the line integral of velocity still should be zero. We can still see that the contribution to the line integral from  $ab$  will be cancelled by the contribution from  $dc$ . But there is no way the contribution to the line integral from  $bd$  is balanced by the contribution

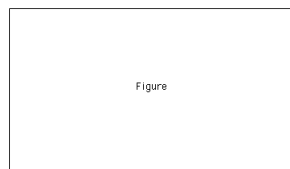


Figure 13.8: (fig:Lec13VortexTube2) A vortex tube ending in the middle of the surrounding surface.

from  $ca$ . The circulation bound by  $bd$  is non-zero, while the circulation bound by  $ca$  is zero. Thus vortex tubes can't end within the fluid.

4. Finally, the 4th law (strength of the vortex tube remains constant in time). This is simply a restatement of Kelvin's theorem:

$$\frac{d\Gamma}{dt} = 0 \quad (13.2)$$

## 13.6 Vorticity equation

Now we're going to derive an equation for the vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \quad (13.3)$$

So to write a vorticity equation we need to cross  $\nabla$  into the momentum equation. Assuming  $\nabla \cdot \mathbf{u} = 0$  (incompressible) and  $\rho = \rho_0$  (barotropic):

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} \quad (13.4)$$

Expanding yields

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \quad (13.5)$$

Here,  $\nu \equiv$  is the kinematic viscosity. Recall from our generation of the Bernoulli functions that we can write

$$\mathbf{g} = -\nabla gz \quad (13.6)$$

and

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) \quad (13.7)$$

$$= \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \boldsymbol{\omega} \quad (13.8)$$

$$= \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) + \boldsymbol{\omega} \times \mathbf{u} \quad (13.9)$$

Substituting this back into the momentum equation and taking the curl yields

$$\nabla \times \left\{ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) + \boldsymbol{\omega} \times \mathbf{u} \right\} = -\nabla gz - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \quad (13.10)$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = -\mathbf{0} - \mathbf{0} + \nabla \times (\nu \nabla^2 \mathbf{u}) \quad (13.11)$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega} \quad (13.12)$$

Here we have made use of a number of facts when applying the curl operator:  $\nabla \times \nabla a = 0$ ,  $\rho = \rho(p)$ , and  $\nabla \times (\nabla^2 \mathbf{a}) = \nabla^2 (\nabla \times \mathbf{a})$ . Also, this assumes  $\nu$  is independent of space. Note  $\nu = \frac{\mu}{\rho}$ , so questionable validity! Let's look into it later in this section.

We can expand the second term on the LHS as follows:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{B} \nabla \cdot \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} \quad (13.13)$$

$$\Rightarrow \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \boldsymbol{\omega} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega} \quad (13.14)$$

$$= \mathbf{0} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \mathbf{0} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} \quad (13.15)$$

$$\Rightarrow \frac{d\boldsymbol{\omega}}{dt} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \nu \nabla^2 \boldsymbol{\omega} \quad (13.16)$$

Here we have made use of incompressibility and the fact that the divergence of the curl of a vector is zero. Thus, by combining the first two terms into a total derivative, the vorticity equation for incompressible barotropic flow becomes:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega \quad (13.17)$$

$$\omega = \nabla \times \mathbf{u} \quad (13.18)$$

Here the first term on the RHS is the change in vorticity from vortex stretching and tilting while the second term is the change from viscous dissipation. The viscous term can be simplified... let's look into it:

$$\nabla \times (\nu \nabla^2 \mathbf{u}) = \nabla \nu \times \nabla^2 \mathbf{u} + \nu \nabla \times \nabla^2 \mathbf{u} \quad (13.19)$$

$$= \nabla \nu \times \nabla^2 \mathbf{u} + \nu \nabla^2 (\nabla \times \mathbf{u}) \quad (13.20)$$

$$= \nabla \frac{\mu}{\rho} \times \nabla^2 \mathbf{u} + \nu \nabla^2 \omega \quad (13.21)$$

For this first term to be small, we have to argue that horizontal  $\rho$  gradients are small when compared to horizontal  $\mathbf{u}$  gradients. The good news is that this is exactly what was argued to allow us to say  $\nabla \cdot \mathbf{u} = 0$ .

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad (13.22)$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \mathbf{u} \nabla \cdot \mathbf{u} = 0 \quad (13.23)$$

For the first term to be small compared to the second term, we need  $\rho$  gradients to be small compared to  $\mathbf{u}$  gradients!

## 13.7 Baroclinic vorticity equation

Now consider the non-barotropic case for the derivation for the vorticity equation starting from the curl of the momentum equation:

$$\nabla \times \left\{ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) + \omega \times \mathbf{u} \right\} = -\nabla g z - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \quad (13.24)$$

Consider  $\nabla \times \frac{1}{\rho} \nabla p$ , where,  $\rho = \rho(p, T, S)$ . Then, with the vector identity:

$$\nabla \times f \mathbf{A} = (\nabla f) \times \mathbf{A} + f (\nabla \times \mathbf{A}) \quad (13.25)$$

$$\Rightarrow \nabla \times \frac{1}{\rho} \nabla p = \nabla \frac{1}{\rho} \times \nabla p + \frac{1}{\rho} \nabla \times \nabla p \quad (13.26)$$

Recalling

$$\frac{d \frac{1}{x}}{dy} = -\frac{1}{x^2} \frac{dx}{dy} \quad (13.27)$$

$$\Rightarrow \nabla \times \frac{1}{\rho} \nabla p = -\frac{1}{\rho^2} \nabla \rho \times \nabla p \quad (13.28)$$

So our vorticity equation becomes:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \omega \quad (13.29)$$

Here, the new term (second on the LHS) is the rate at which vorticity changes due to baroclinicity (see figure 13.9).

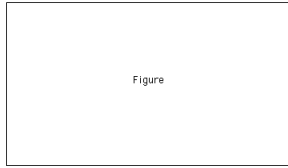


Figure 13.9: (fig:Lec13Baroclinic) Vorticity in a baroclinic situation. Note that the term gets larger the more... **(Jim – The rest got cut off)**.

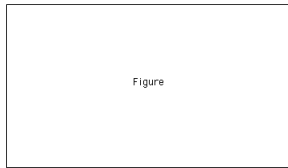


Figure 13.10: (fig:Lec13VortexStretching)  $\omega'_x \geq \omega_x$  through conservation of angular momentum  $\Rightarrow \frac{D\omega_x}{Dt}$  increasing.

### 13.8 More on the barotropic vorticity equation

Back to the barotropic vorticity equation:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega \tag{13.30}$$

Let's concentrate on the first term of the RHS:

$$(\omega \cdot \nabla) \mathbf{u} = \left( \omega_x \frac{d}{dx} + \omega_y \frac{d}{dy} + \omega_z \frac{d}{dz} \right) \mathbf{u} \tag{13.31}$$

Just consider the  $x$  bit:

$$\left( \omega_x \frac{du}{dx} + \omega_y \frac{du}{dy} + \omega_z \frac{du}{dz} \right) \mathbf{i} \tag{13.32}$$

Here the first term is the stretching terms and the next two are the tilting terms. The stretching term changes vorticity through conservation of angular momentum (see figure 13.10). The tilting terms change vorticity by tilting vorticity vectors into new directions (see figure 13.11).

With the barotropic vorticity equation, we went from (momentum)

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \tag{13.33}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{13.34}$$

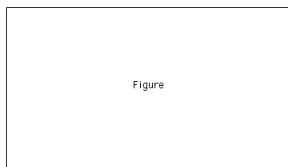


Figure 13.11: (fig:Lec13VortexTilting) The  $\frac{\partial u}{\partial y}$  tilts the initial  $\omega_y$  vector so that the new  $\omega'$  has a projection into the  $x$ -direction,  $\omega'_x \Rightarrow \frac{D\omega_x}{Dt}$  increasing. **(Q4)**

to (vorticity)

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega \quad (13.35)$$

$$\omega = \nabla \times \mathbf{u} \quad (13.36)$$

Going to vorticity allowed us to get rid of pressure and gravity. Now consider inviscid:

$$\mu = 0 \Rightarrow \nu = 0 \quad (13.37)$$

$$\frac{Dw}{Dt} = \omega \cdot \nabla \mathbf{u} \quad (13.38)$$

Assume 2D (this is a very strong constraint)

$$\omega = 0\mathbf{i} + 0\mathbf{j} + \omega_z\mathbf{k} \quad (13.39)$$

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \quad (13.40)$$

Thus

$$x : \frac{D\omega_x}{Dt} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} = 0 \quad (13.41)$$

$$y : \frac{D\omega_y}{Dt} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} = 0 \quad (13.42)$$

$$z : \frac{D\omega_z}{Dt} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} = 0 \quad (13.43)$$

So we have that  $\omega$  is constant following the flow. That is, for 2D barotropic, inviscid flow

$$\frac{D\omega}{Dt} = 0 \quad (13.44)$$

Imagine  $\omega = 0$  at  $t = 0$ , then  $\omega = 0$  for all time. This implies that if we have a 2D object initially at rest in an irrotational flow, when we start to move the object the flow should remain irrotational.

$$\omega_o = 0, \quad \frac{D\omega}{Dt} = 0 \quad (13.45)$$

**(Jim – You seem to be repeating yourself a little bit here at the end).**

## 13.9 Midterm review

- Definition of a fluid
- Difference between Eulerian and Lagrangian
- Produce the material derivative (and understand what it means!)
- Cauchy-Stokes
  - Linear strain rate
  - Shear strain rate
  - Rigid rotation rate
- How the bits of the velocity gradient tensor deform a fluid blob
- Intensive versus extensive properties



- Know and understand the various flavors of the Reynold's transport theorem
- Understand how to derive various equations of motion using the integral form
  - Mass:  $\frac{dM}{dt} = 0$
  - Momentum:  $\frac{dP}{dt} = \mathbf{F}_B + \mathbf{F}_S + \mathbf{F}_L$
  - 1st Law:  $\frac{dE}{dt} = \dot{Q} - \dot{W}$

You won't be asked to derive the nasty viscosity bits, but you will be expected to understand them.

- "Understand" the Boussinesq approximation
- Know and understand all equations (N-S, Boussinesq, Euler)
- Be able to produce Bernoulli functions and know where they are constant
- Potential flow
  - Streamfunction
  - Velocity potential
  - Laplace equations
  - Dirichlet/Neumann
- Circulation and Stoke's theorem – use on SBR and point vortex
- Kelvin's Circulation Theorem and the Helmholtz Vortex Theorems
- Derivation and understanding of the vorticity equations