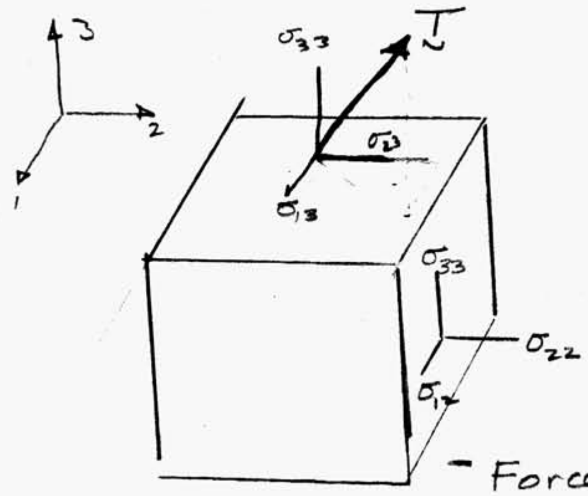
# Stress




Traction $\tilde{T}$ on material may depend on area

Body forces $\tilde{g}$
depend on volume
neglect in this discussion

- Forces inside body which are a reaction to traction are stresses
- Homogeneous if forces acting on a surface of fixed shape and orientation do not depend on position in body.

Then traction on each face of unit cube can be characterized.

For body to be in static equilibrium, Tractions on (+2 face) = (−2 face) and opposite in direction otherwise net unbalance causes acceleration governed by Newton's law.

Define stress on each face as traction / area

$$\ni: \tilde{T} = \underset{\approx}{\sigma}\,\tilde{A}$$ i.e. stress tensor relates area normal $\rightarrow$ traction

$$\boxed{T_i = \sigma_{ij} A_j}$$

$\sigma_{ij} \triangleq T_i$ on $j$ face

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normal components, $\sigma_{ii}$ + if traction

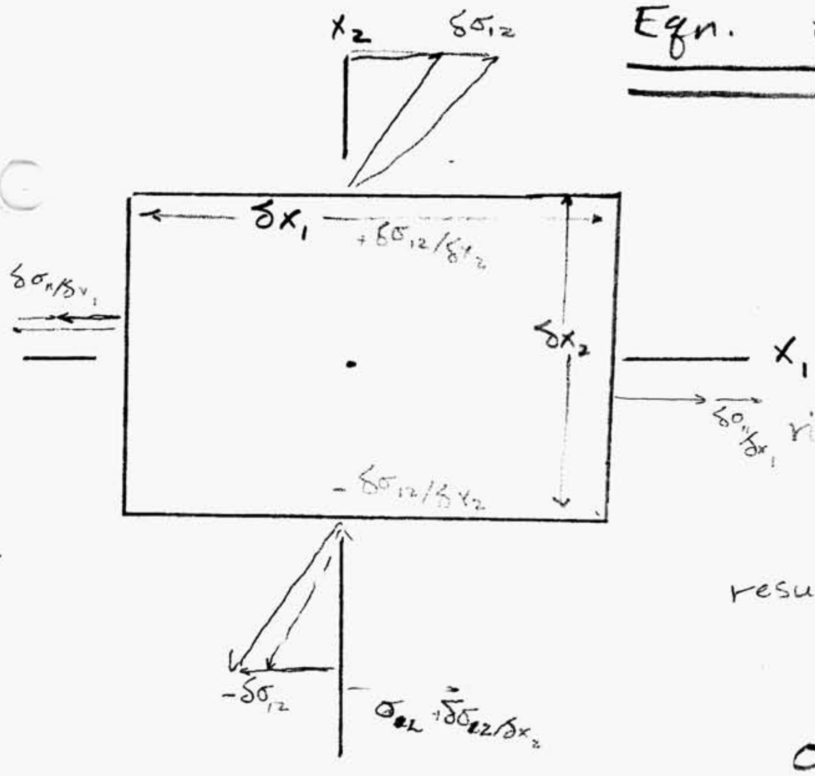
shear components $\sigma_{ij},\ i \neq j$ + if in positive dir on " face

+ if in negative dir on " face

$\sigma_{11}$ $\sigma_{12}$ $\sigma_{13}$ shear

$\sigma_{21}$ $\sigma_{22}$ $\sigma_{23}$ normal

# Eqn. Equilibrium




On $x_1$ face in $x_1$ direct

left $\sigma_{11}\,\delta x_2 \delta x_3 + \frac{\partial\sigma_{11}}{\partial x_1}\frac{\delta x_1}{2}\partial x_2\partial x_3$

right $-\left[\sigma_{11}\,\delta x_2 \partial x_3 - \frac{\partial \sigma_{11}}{\partial x_1}\frac{\delta x_1}{2}\delta x_2 \partial x_3\right]$

result. $\frac{\delta\sigma_{11}}{\partial x_1}(\partial x_1\partial x_2\partial x_3)$

On $x_2$ in $x_1$ direction

top $\sigma_{12}(\partial x_1 \partial x_3) + \frac{\partial\sigma_{12}}{\partial x_2}\frac{\partial x_2}{2}(\partial x_1\partial x_3)$

bottom $-\left[\sigma_{12}(\partial x_1\partial x_3) + -\frac{\partial\sigma_{12}}{\partial x_2}\left(\frac{\partial x_2}{2}\right)(\partial x_1 \partial x_3)\right]$

$$\frac{\partial\sigma_{12}}{\partial x_2}(\partial x_2\ \partial x_1\partial x_3)$$

By Newton's law, sum of forces in $x_1$ direction equals mass/vol x acceleration in that direction

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3}\left(+\rho g_1\right) = \rho\ddot{x}_1$$

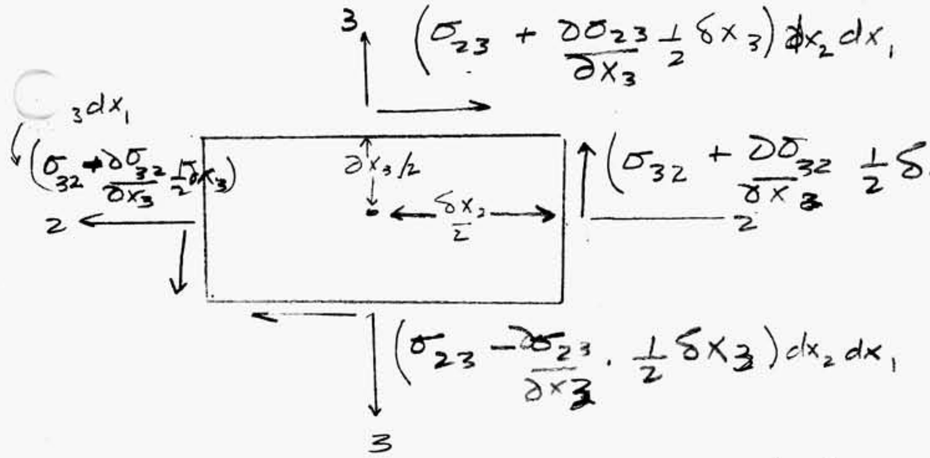
In ~~general~~,

$$\boxed{\frac{\partial\sigma_{ij}}{\partial x_j} + \rho g_i = \rho\ddot{x}_i}$$

If $\ddot{x}_i = 0$, i.e. in static eq.

$$\boxed{\frac{\partial\sigma_{ij}}{\partial x_j} + \rho g_i = 0}$$

# Shear moments and symmetry




2 face is $\frac{1}{2} \delta x_3$ moment arm for

moments are anticlockwise

moment arm for 3 face is $\frac{1}{2} \delta x_3$
moments are clockwise

$$2\sigma_{32} \frac{dx_3}{2} (dx_2 \, dx_1) - 2\sigma_{23} \frac{dx_2}{2} dx_3 \, dx_1 + G_1 \, dx_1 \, dx_2 \, dx_3 = I_1 \ddot{\theta}_1$$

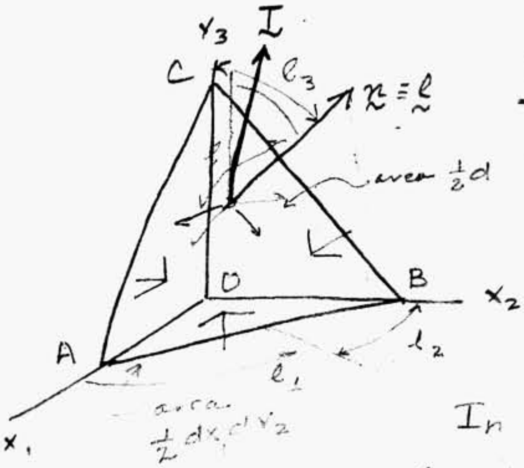
but assume no body torques $G_1 = 0$
and note $I_1$ order of mag $\rho \delta x^5 \to 0$ as $dx \downarrow$

then $(\sigma_{32} - \sigma_{23}) \, dx_1 \, dx_2 \, dx_3 = 0$

$$\Rightarrow \sigma_{32} = \sigma_{23}$$

or $\sigma_{ij} = \sigma_{ji}$

$\Rightarrow$ Stress tensor is symmetric.




## Proof that stress is a tensor

### Cauchy tetrahedron

Total force on face with area ABC

$$\underset{\sim}{T} \stackrel{\Delta}{=} \underset{\sim}{p} \cdot (ABC)$$

In $x_1$ direction

$$p_1 (ABC) = \sigma_{11} (BOC) + \sigma_{12} (AOC) + \sigma_{13} (AOB)$$

$$p_1 = \sigma_{11} \ell_1 + \sigma_{12} \ell_2 + \sigma_{13} \ell_3$$

where $\ell_1 = \frac{AOC}{ABC} = \angle x_1 \underset{\sim}{\ell}$

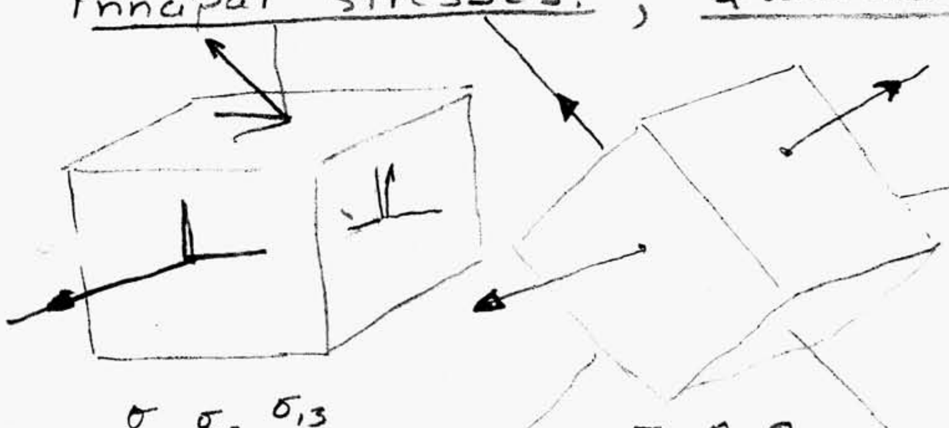
sim. $p_2 = \sigma_{21} \ell_1 + \sigma_{22} \ell_2 + \sigma_{23} \ell_3$

$p_3 = \sigma_{31} \ell_1 + \sigma_{32} \ell_2 + \sigma_{33} \ell_3$

$$\boxed{\text{or } p_i = \sigma_{ij} \ell_j \quad \text{i.e. tensor transformation rule}}$$

⇒ Stress is a 2nd rank, symmetric tensor

⇒ Principal stresses, Quadric



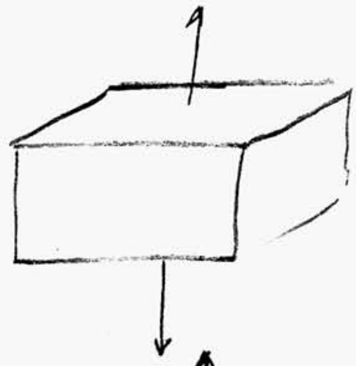
$$\begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix} \longrightarrow \begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{matrix}$$

# Special Forms of Stress 

i.) uniaxial stress

$$\begin{matrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Rod stretche



ii.) biaxial stress

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

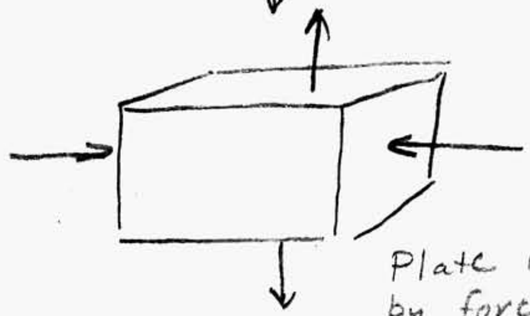
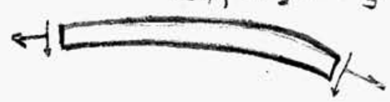


Plate loaded by forces & coupl at end $\Delta x_1, \Delta x_2 \ll x_3$

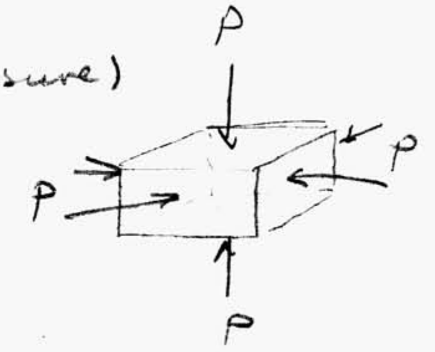


iii) triaxial stress

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{matrix}$$

iv) hydrostatic stress (pressure)

$$\begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} = -p\delta_{ij}$$




v.) Pure shear

$$\begin{bmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

see above

vi.) Simple shear

$$\begin{matrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

# Summary : Stress Tensor

1.) Traction on a plane with direction cosine $\underset{\sim}{l}$

$$\underset{\sim}{T} = \underset{\approx}{\sigma}\ \underset{\sim}{l} \qquad T_i = \sigma_{ij}\, l_j$$

2.) Eqns of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j = m\ddot{x}_j$$

3.) Stress is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

4.) Principal stress directions
values

$$\begin{matrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{matrix}$$

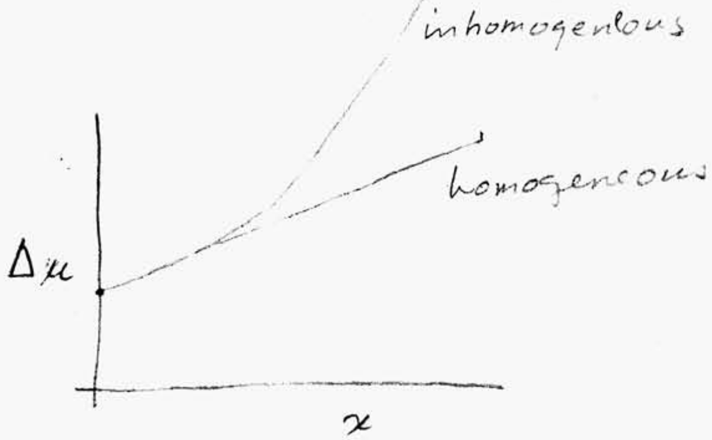
5.) Stress quadric construction

# Stress, Strain, Elasticity:

Nye, Chap 5 & 6, pps 82 – 105.

## 1. One dimensional strain

Relative displacements important


<u>Strain</u> at a point P is

$$\epsilon = \frac{\text{increase in length}}{\text{original length}} = \frac{P'Q' - PQ}{PQ} = \frac{\Delta u}{\Delta x}$$

(slope of curve above)

**<u>Goal</u>:** For a given deformation of body, Define a tensor that describes the change of direction and length of any vector in body.

Want to map vector in body (undef) into vector in body (def.)