## Elasticity of Rocks

### *Assigned Reading:*

Gueguen, Yves and V. Palcisukas, *Introduction to the Physics of Rocks,* Chapter III, p. 53-61; Chapter IV. 73-92. (Essentially a discussion of some of the material we have covered over the last couple of week. I hope this reading is easier no than it would have been before we started.)+

### *Resource reading:*

Atkinson BK (1987) Fracture Mechanics of Rock. Academic Press, London UK, pp 534 Chapters 1&11

Johnson AM (1970) Physical Processes in Geology. Freeman Cooper, San Francisco CA, Chapters 9-11.

Clyne, T. W., and P. J. Withers, *An Introduction to Metal Matrix Composites,* Cambridge University Press, 1993.

Hearmon, R. F. S., *An Introduction to Applied Anisotropic Elasticity,* Oxford University Press, 1961.

Mavko, G., T. Mukerji, and J. Dvorkin, *The Rock Physics Handbook,* Cambridge Univ. Press, 1998.

Watt, J. P., G. F. Davies, and R. J. O'Connell, "The Elastic Properties of Composite Materials", *Rev. Geophys. Space Phys.,* 14, 541-563, 1976.

Muskhelishvili, N. I.(Nikolaĭ Ivanovich),1891- Some basic problems of the mathematical theory of elasticity; fundamental equations, plane theory of elasticity, torsion, and bending, Groningen, P. Noordhoff, 1963. QA931.M9871 1954

## Exact Elastic Treatment of Simple Geometry

### *Fundamental Equations of Elasticity:*

Equilibrium, Compatibility, Constitutive Law
Biharmonic Equation, Airy Stress Function

### *Particular Solutions*

Internally Pressurized Cylinder
Externally Stressed Hole
Externally Stressed Elliptical Hole

### *Elastic Moduli of Cracked Solids*

## Basic Equations of Isotropic Elasiticity

### *Elasticity of isotropic elastic materials*

$$\boxed{\varepsilon_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}} \qquad \boxed{\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}}$$

### *Equilibrium (Static)*

$$\boxed{\frac{\partial\sigma_{ij}}{\partial x_j} = 0}$$

### *Compatibility*

$$\boxed{\frac{\partial^2\varepsilon_{ij}}{\partial x_k\partial x_l} + \frac{\partial^2\varepsilon_{kl}}{\partial x_i\partial x_j} - \frac{\partial^2\varepsilon_{ik}}{\partial x_j\partial x_l} - \frac{\partial^2\varepsilon_{jl}}{\partial x_i\partial x_k} = 0}$$

Proof:

Strain may be viewed as the differential of the displacement change vector. Provided that no gaps open up and that no overlaps develop, the displacements must be continuous.

If the displacement change vector, $\mathbf{\Delta u} = [u_1, u_2, u_3]$, then

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \textit{and} \quad \frac{\partial^2\varepsilon_{11}}{\partial^2 x_2} = \frac{\partial^3 u_1}{\partial^2 x_2\partial x_1}$$

calculate $\frac{\partial^2\varepsilon_{22}}{\partial^2 x_1}$ and $\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$. Notice that $\frac{\partial^2\varepsilon_{22}}{\partial^2 x_1} + \frac{\partial^2\varepsilon_{11}}{\partial^2 x_2} = 2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$

*In general any problem for the statics of a three dimensional body loaded externally and with no body forces is well-posed using these equations as long as sufficient boundary conditions are given.*

## Consider plane strain and plane stress (i.e. 2-D) without body forces:

*Equilibrium*

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

*Compatibility*

$$\frac{\partial^2 \varepsilon_{22}}{\partial^2 x_1} + \frac{\partial^2 \varepsilon_{11}}{\partial^2 x_2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

*Elasticity*

$$\varepsilon_{11} = \frac{(1+\nu)}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = \frac{\sigma_{11}}{E} - \frac{\nu\sigma_{22}}{E}$$

$$\varepsilon_{12} = \frac{(1+\nu)\sigma_{12}}{E}$$

*Equilibrium, Compatibility and Elasticity require*

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\right)(\sigma_{11} + \sigma_{22}) = \boxed{0 = \nabla^2 \sigma_{ii}}$$

## Airy Stress Function:

Assume that there exists a function, $\Phi$, the Airy stress function, such that

$$\frac{\partial^2 \Phi}{\partial x_2{}^2} = \sigma_{11}$$

$$\frac{\partial^2 \Phi}{\partial x_1{}^2} = \sigma_{22}$$

$$-\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = \sigma_{12}$$

Then from above,

$\boxed{\nabla^4 \Phi = 0}$ Biharmonic Equation

*Biharmonic Equation: Method of solution is 1.)Find a general Airy's stress function with appropriate symmetry; 2.) Use differential equations to get stress; 3.) Use boundary conditions to find arbitrary constants: 4.) Use Hook's law to get strains; 5.) Integrate to get displacements*

## Elasticity in Problems with Cylindrical and Spherical Symmetry

## In problems with cylindrical symmetry, *r, θ, z:*

*Strain:*

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right), \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right), \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

The gradient operator and the Laplacian in cylindrical coordinates are

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

*Equilibrium:*

In problems with cylindrical symmetry, *r, θ, z:*

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} = 0$$

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = 0$$

***Spherical symmetry, r, θ, ϕ,***

*Special case that displacements occur in the radial direction only:*

*Strain:*

In problems with spherical symmetry, *r, θ, ϕ,* for the special case that displacements occur in the radial direction only*:*

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u_r}{r}, \qquad \varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{rz} = 0$$

*Equilibrium:*

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\left(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)=0, \qquad \frac{\partial \sigma_{\theta\theta}}{\partial \theta}=\frac{\partial \sigma_{\phi\phi}}{\partial \phi}=0, \qquad \sigma_{\theta\theta}=\sigma_{\phi\phi}$$

## Example 1: Stress around a circular pipe (tube)

Assume plane strain and plane stress, that the pipe is internally pressured with no applied stresses.

*New boundary conditions are*

$\sigma_{rr}$=-P and $\sigma_{r\theta}$=0 at $r=r_o$

$\sigma_{rr}=\sigma_{r\theta}=0$ at $r=\infty$

Notice that Φ must be independent of θ.

Now, let $r=e^t$

$\Phi=e^{mt}$ and substitute into the biharmonic.

A general solution is

$\Phi=A+Br^2+C\bullet\ln(r)+Er^2\bullet\ln(r)$

Check to see if Φ satisfies the biharmonic

$$\frac{\partial^2}{\partial r^2}\left[\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial^2\Phi}{\partial r^2}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right]=\frac{\partial^4\Phi}{\partial r^4}+\frac{1}{r}\frac{\partial^3\Phi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r^3}\frac{\partial\Phi}{\partial r}$$

Now

$$\frac{\partial\Phi}{\partial r}=\frac{C}{r}+2Br+Er\bullet\ln(r)+Er$$

$$\frac{\partial^2\Phi}{\partial r^2}=-\frac{C}{r^2}+2B+E\bullet\ln(r)+3E$$

$$\frac{\partial^3\Phi}{\partial r^3}=\frac{2C}{r^3}+2Br+Er\bullet\ln(r)+Er$$

$$\frac{\partial^4\Phi}{\partial r^4}=-\frac{6C}{r^4}-\frac{2E}{r^2}$$

so...

$$\nabla^4\Phi = \left[-\frac{6C}{r^4} - \frac{2E}{r^2}\right] + \frac{1}{r}\left[\frac{2C}{r^3} + 2\frac{E}{r}\right] + \frac{1}{r^2}\left[-\frac{C}{r^2} + 2B + E \bullet \ln(r) + 3E\right]$$
$$-\frac{1}{r^3}\left[\frac{C}{r} + 2Br + Er \bullet \ln(r) + Er\right]$$
$$and \Rightarrow \nabla^4\Phi = 0$$

Now solve for the constants in the solution using the boundary conditions

$$\sigma_{rr} = \frac{1}{r}\left(\frac{\partial\Phi}{\partial r}\right)$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(A + Br^2 + C\ln(r) + Er^2\ln(r)\right) = 2B + \frac{C}{r^2} + 2E\ln(r) + E$$
$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = 2B - \frac{C}{r^2} + E(3 + 2\ln(r))$$

As $r \to \infty, \sigma_{rr}$ , $\sigma_{\theta\theta}$=0.
So E, B=0.

At $r$=$r_0$ $\sigma_{rr}$ =-P

$$\sigma_{rr}\,|_{r=r_o} = \frac{C}{r_o^{\,2}} = -P$$
$$\sigma_{rr} = -P\frac{r^2}{r_o^{\,2}}$$
$$\sigma_{\theta\theta} = -P\frac{r_o^{\,2}}{r^2}$$
$$\sigma_{r\theta} = 0$$

*.Remarks: Stress falls off as $r^2$, depends linearly on P*
*Hoop stresses are tensile.*
*If fractures occur they occur along planes of maximum tensile stress.*

## Example 2: Stress and Strain Concentration around a Spherical Hole loaded by a pressure at infinity:

### *Strain about a spherical hole.*

Suppose that a body loaded by pressure $\overline{\sigma}$ contains a spherical hole. By symmetry we suppose that the displacements must be a function of r only. Then guess that

$$u_r = ar + \frac{b}{r^2}$$

## *Pore Compressibility:*

Suppose we consider a spherical hole with an pressure $\overline{\sigma}$ applied at infinity.The material is elastic. From symmetry we suppose that the strain will be in the radial direction only, and that it will be proportional to the applied pressure:

$$\varepsilon_{rr} = \frac{dR}{R} = C \cdot d\overline{\sigma} \quad \textit{where } C = \frac{1}{K_0}\frac{(1-\nu)}{2(1-2\nu)}$$

Then the change in pore volume is $dV_p = 4\pi R^2 dR = 3V_p \dfrac{dR}{R}$

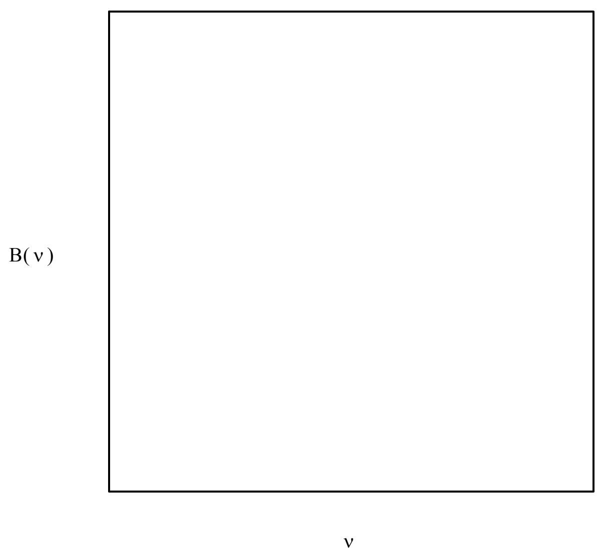
So the volumetric strain, $\varepsilon_V$ is

$$\varepsilon_V = \frac{3(1-\nu)}{2K_0(1-2\nu)} d\overline{\sigma}$$

and the pore stiffness is

$$\frac{dV_p}{V_p d\overline{\sigma}} = \boxed{\frac{3(1-\nu)}{2K_0(1-2\nu)} = \frac{1}{K_\phi}}$$

If the Poisson's ratio is 0.25, then $1/K_\phi$ is 2.25; as the solid material becomes incompressible the pore compressibility becomes large.



## Stress concentrations around pores, flaws, and inclusions:

| | |
|---|---|
| Cylindrical hole | Internal P |
| “ | External P |
| “ | Uniaxial load, σ |
| | |
| Spherical hole | Internal P |
| “ | External P |
| | Uniaxial load, σ |
| Elliptical hole | External P |
| “ | Biaxial Load, S |