# 12.520 Lecture Notes 25

## The Stream Function

For continuum mechanics in general and fluid mechanics specifically, a number of "laws" are expressed in terms of differential equations. For example,

1) Newton's second law (F = ma) (general)

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{Dv_i}{Dt}$$

2) Rheology (constitutive equation) (Newtonian fluid)

$$\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{\varepsilon}_{ij}$$

3) Definition of strain rate (general)

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

4) Continuity (conservation of mass) (incompressible)

$$\frac{\partial v_i}{\partial x_i} = 0$$

These 4 coupled first order differential equations, plus boundary conditions, can be solved to determine fluid flow for a variety of interesting applications.

Alternatively, they can be combined to form a single fourth order differential equation.

For fluids, this fourth order equation often involves the <u>stream</u> <u>function</u>.

Consider a 2-D flow with velocities $v_1$, $v_3$ in the $x_1$, $x_3$ plane ($v_2 = 0$)

If $v_1 = -\frac{\partial \Psi}{\partial x_3}$

$$v_3 = \frac{\partial \Psi}{\partial x_1} \quad \Rightarrow \quad \nabla \cdot \underset{\sim}{v} = \frac{\partial v_i}{\partial x_i} = -\frac{\partial^2 \Psi}{\partial x_1 \partial x_3} + \frac{\partial^2 \Psi}{\partial x_3 \partial x_1} = 0$$

Incompressibility is automatically satisfied!

[In general, if $\underset{\sim}{v} = \nabla \times \underset{\sim}{\Psi}$, $\nabla \cdot \underset{\sim}{v} = 0$. Here $\underset{\sim}{\Psi} = (0, \Psi, 0)$]

Substituting into the (steady) Navier-Stokes equation

$$-\frac{\partial p}{\partial x_1} - \eta \left( \frac{\partial^3 \Psi}{\partial x_1{}^2 \partial x_3} + \frac{\partial^3 \Psi}{\partial x_3{}^3} \right) + \rho f_1 = 0$$

$$-\frac{\partial p}{\partial x_3} + \eta \left( \frac{\partial^3 \Psi}{\partial x_1{}^3} + \frac{\partial^3 \Psi}{\partial x_1 \partial x_3{}^2} \right) + \rho f_3 = 0$$

Now take $\frac{\partial}{\partial x_3}$ of first, $\frac{\partial}{\partial x_1}$ of second

$$-\frac{\partial^2 p}{\partial x_1 \partial x_3} - \eta \left( \frac{\partial^4 \Psi}{\partial x_1{}^2 \partial x_3{}^2} + \frac{\partial^4 \Psi}{\partial x_3{}^4} \right) + \rho \frac{\partial f_1}{\partial x_3} = 0$$

$$-\frac{\partial^2 p}{\partial x_1 \partial x_3} + \eta \left( \frac{\partial^4 \Psi}{\partial x_1{}^4} + \frac{\partial^4 \Psi}{\partial x_1{}^2 \partial x_3{}^2} \right) + \rho \frac{\partial f_3}{\partial x_1} = 0$$

Subtract:

$$\eta \left( \frac{\partial^4 \Psi}{\partial x_1{}^4} + 2 \frac{\partial^4 \Psi}{\partial x_1{}^2 \partial x_3{}^2} + \frac{\partial^4 \Psi}{\partial x_3{}^4} \right) + \rho \left( \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) = 0$$

$$\frac{\partial^4 \Psi}{\partial x_1{}^4} + 2 \frac{\partial^4 \Psi}{\partial x_1{}^2 \partial x_3{}^2} + \frac{\partial^4 \Psi}{\partial x_3{}^4} = \nabla^2 (\nabla^2 \Psi) = \nabla^4 \Psi$$

$\nabla^4$ is called biharmonic operator.

For uniform or no $\underset{\sim}{f}$: $\quad \nabla^4 \Psi = 0$

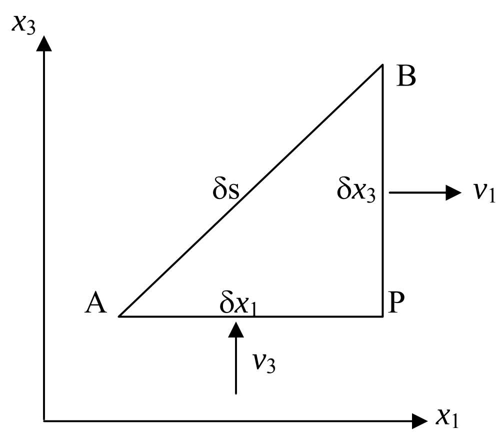
Advantages of using the biharmonic operator are

1. only one equation
2. efficient solution

Disadvantage: Loss of "physical insight".

**Physical Interpretation of Stream Function**

Consider triangle APB.




For incompressible fluid,

$$\text{flux}_{AP} + \text{flux}_{BP} + \text{flux}_{AB} = 0$$

$$-v_3\delta x_1 + v_1\delta x_3 + \text{flux}_{AB} = 0$$

$$\text{flux}_{AB} = v_3\delta x_1 - v_1\delta x_3 = \frac{\partial\Psi}{\partial x_1}\delta x_1 + \frac{\partial\Psi}{\partial x_3}\delta x_3 = \delta\Psi$$

$$\text{or } \int_A^B d\Psi = \Psi_B - \Psi_A$$

Difference in Ψ represents the flux crossing the curve.

**Solution of biharmonic**

Polynomials (e.g., for Conette flow, $\Psi = -\frac{v_0 x_3^{\;2}}{2h}$)

Separation of variables:

$$\Psi = X(x)Z(z)$$

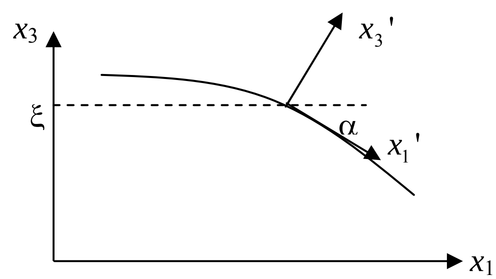
$$\nabla^4\Psi = 0 \Rightarrow X''''Z + 2X''Z'' + XZ'''' = 0$$

$$\frac{X''''}{X} + 2\frac{X''}{X}\frac{Z''}{Z} + \frac{Z''''}{Z} = 0$$

Harmonic $\Psi = \sin\frac{2\pi x}{\lambda} Z(z)$

Solution: $\Psi = [(A + Bz)\exp(\frac{2\pi z}{\lambda}) + (C + Dz)\exp(-\frac{2\pi z}{\lambda})]\sin(\frac{2\pi x}{\lambda})$

Physical boundary conditions: $T_n = 0$ $T_\tau = 0$




In $x_1', x_3'$ coordinates, at $x_3 = \xi(x_1)$:

$$\sigma_{3'3'} = 0$$

$$\sigma_{3'1'} = \sigma_{1'3'} = 0$$

Have solution to biharmonic in terms of $x_1, x_3$ -- easily applied at $x_3 = 0$.

Need to take physical ($x_1', x_3'$) boundary conditions and

1. rotate to $x_1, x_3$ space
2. Taylor's series expansion
3. subtract out hydrostatic reference state

   Result (to first order in $\xi / \lambda$)

$$\sigma = \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & \rho g \xi \end{pmatrix}$$

4. solve biharmonic.

# Postglacial Rebound

### Decay of Boundary Undulations (1/2 space, uniform $\eta$)

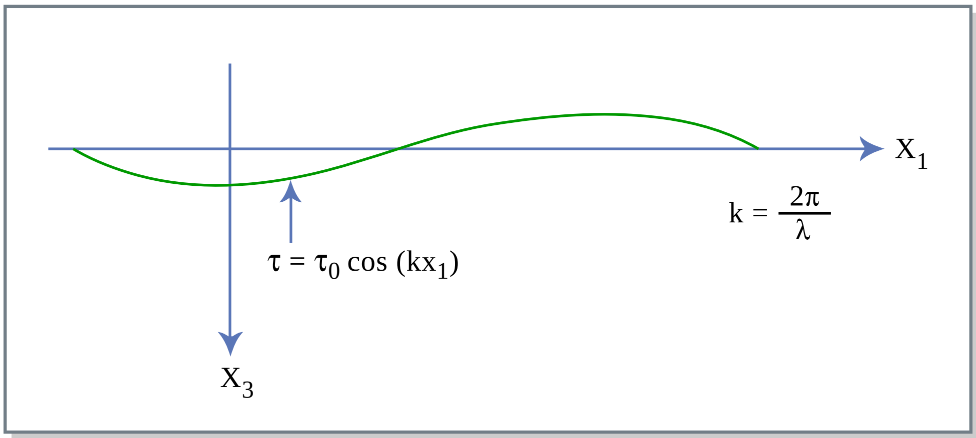



Figure 25.1

Figure by MIT OCW.

- Assume uniform $\eta$
- Subtract out lithostatic pressure $P = p - \rho g x_3$
- Assume $\rho g$ uniform
- Use stream function $\Psi$

$$v_1 = -\frac{\partial \Psi}{\partial x_3} \qquad v_3 = \frac{\partial \Psi}{\partial x_1}$$

$$\Rightarrow \nabla^4 \Psi = 0$$

Solution: $\Psi = \left[(A + Bkx_3)\exp(-kx_3) + (C + Dkx_3)\exp(kx_3)\right] \cdot \sin kx_1$

Boundary conditions:

at $x_3 = 0$ (mathematical, not physical)

$$\sigma_{33} = \rho g \zeta$$

$$\sigma_{13} = 0 = \eta\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)$$

at $x_3 \to \infty$, must be bounded

$$\Rightarrow C = D = 0$$

In order that $\sigma_{13} = 0$ at $x_3 = 0$,

$$-\frac{\partial^2 \Psi}{\partial {x_3}^2} + \frac{\partial^2 \Psi}{\partial {x_1}^2} = 0$$

$$\Rightarrow B = A$$

$$\text{or } \Psi = A(1 + kx_3)\exp\left(-kx_3\right) \cdot \sin kx_1$$

Then

$$v_1 = Ak^2 x_3 \exp\left(-kx_3\right) \cdot \sin kx_1$$

$$v_3 = Ak(1 + kx_3)\exp\left(-kx_3\right) \cdot \cos kx_1$$

$$\text{at } x_3 = 0 \quad v_3 = \dot{\zeta} = Ak\cos(kx_1)$$

Now

$$\sigma_{33} = -p + 2\eta\dot{\varepsilon}_{33}$$

$$\dot{\varepsilon}_{33} = 0 \quad \text{at } x_3 = 0$$

To get $p$, use $-\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho x_1 = 0$

for $i = 1$

$$\Rightarrow \; -\frac{\partial p}{\partial x_1} + \eta\left(\frac{\partial^2 v_1}{\partial {x_1}^2} + \frac{\partial^2 v_1}{\partial {x_3}^2}\right) = 0$$

Substitute for $v_1$ and integrating $\Rightarrow p\big|_{x_3=0} = 2\eta k^2 A\cos kx_1$

But $p = -\rho g\zeta \Rightarrow A = -\frac{\rho g \zeta_0}{2k^2\eta}$

Or $\dot{\zeta}_0 = -\frac{\rho g \zeta_0}{2k\eta} = -\frac{\rho g \lambda \zeta_0}{4\pi\eta}$

Or $\zeta_0 = \zeta_0\big|_{t=0} \exp\left(-\frac{\rho g t}{2k\eta}\right) = \zeta_0\big|_{t=0} \exp\left(-\frac{t}{\tau}\right)$

where $\tau = \frac{2k\eta}{\rho g} = \frac{4\pi\eta}{\rho g \lambda}$

Solving for $\eta$: $\quad \eta = \frac{\rho g \lambda \tau}{4\pi}$

For curves shown,

$$\left.\begin{array}{l}\tau \sim 5000 \text{ yr} \\ \lambda \sim 3000 \text{ km}\end{array}\right\} \Rightarrow \eta \sim 10^{21} \text{ Pa}$$

Note: stream function $\sim \exp(-kx_3) = \exp(-\frac{2\pi x_3}{\lambda})$

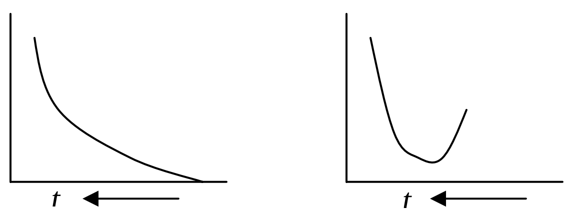
Falls off to $\sim 1/e$ at $x_3 \sim \frac{\lambda}{2\pi}$

Senses to fairly great depth

$\Rightarrow$ postglacial rebound doesn't reveal the details of mantle viscosity structure, but only the gross structure.

Note: Behavior at Hudson Bay and Boston different:

Hudson Bay　　　　Boston

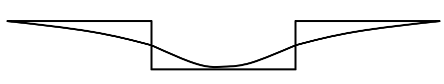



Continuous uplift　　　　Subsidence, then uplift

Is this consistent with uniform 1/2 space?

$$\tau = \frac{4\pi\eta}{\rho g \lambda}$$

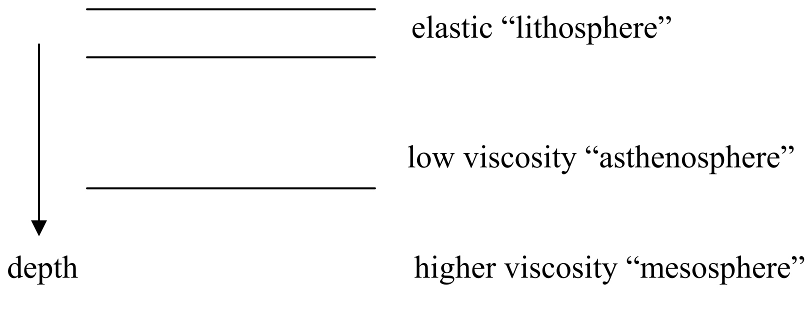
Decompose into Fourier components



$\Rightarrow$ ?

Details depend on geometry of ice load and elastic support of load.

Suppose we require faster relaxation for short λ than for long λ.




How to get solution? What are the boundary conditions?