12.005 Lecture Notes 3

Tensors

Most physical quantities that are important in continuum mechanics like temperature, force, and stress can be represented by a tensor. Temperature can be specified by stating a single numerical value called a scalar and is called a zeroth-order tensor. A force, however, must be specified by stating both a magnitude and direction. It is an example of a first-order tensor. Specifying a stress is even more complicated and requires stating a magnitude and two directions—the direction of a force vector and the direction of the normal vector to the plane on which the force acts. Stresses are represented by second-order tensors.

Stress Tensor

Representing a force in three dimensions requires three numbers, each referenced to a coordinate axis. Representing the state of stress in three dimensions requires nine numbers, each referenced to a coordinate axis and a plane perpendicular to the coordinate axes.

Returning to determining traction vectors on arbitrary surfaces.

Consider two surfaces S_1 and S_2 at point Q.

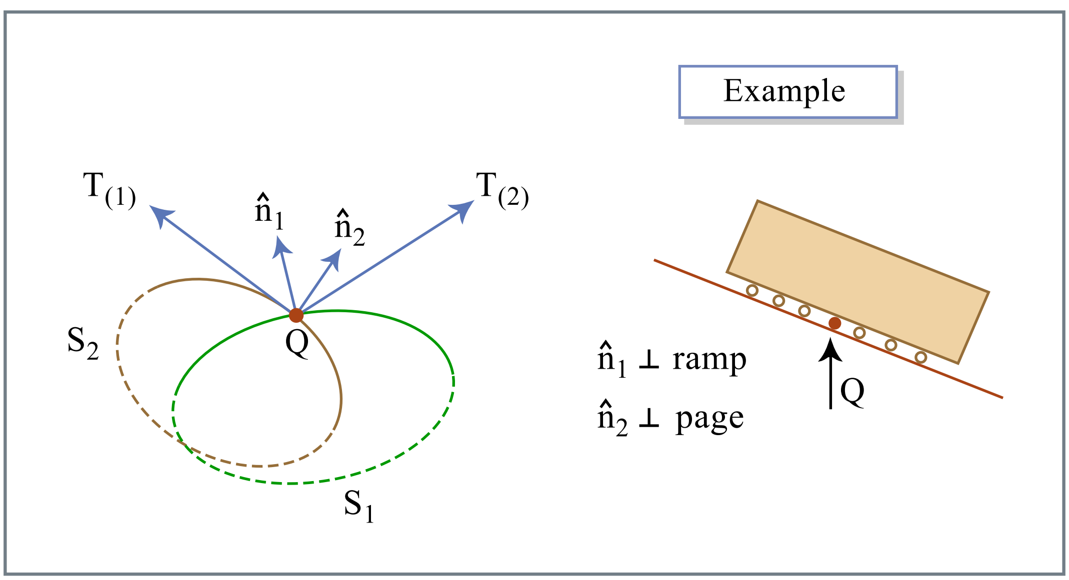

Figure 3.1

Figure by MIT OCW.

Tractions at a point depend on the orientation of the surface.

How to determine $\underset{\sim}{T}$, given $\hat{n}$?

For special cases $\hat{n}$ along axes.

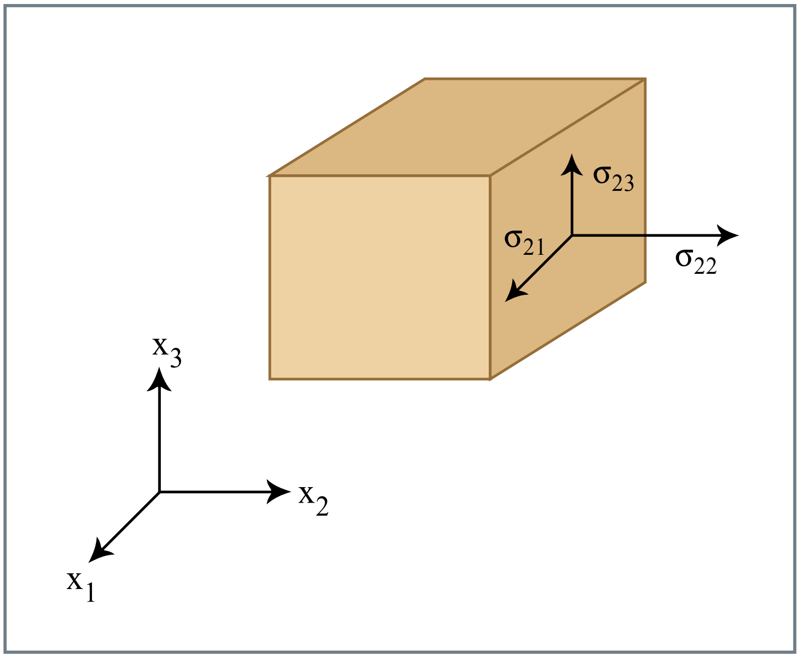

Figure 3.2

Figure by MIT OCW.

In vector notation, the tractions on the faces of the cube are written:

$$T_{(1)} = \langle \sigma_{11}, \sigma_{12}, \sigma_{13} \rangle$$
$$T_{(2)} = \langle \sigma_{21}, \sigma_{22}, \sigma_{23} \rangle$$
$$T_{(3)} = \langle \sigma_{31}, \sigma_{32}, \sigma_{33} \rangle$$

In matrix notation, the tractions are written:

$$\begin{pmatrix} T_{(1)} \\ T_{(2)} \\ T_{(3)} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

This matrix is generally referred to as the stress tensor. It is the complete representation of stress at a point.

The Cauchy Tetrahedron and Traction on Arbitrary Planes

The traction vector at a point on an arbitrarily oriented plane can be found if $T_{(1)}, T_{(2)}, T_{(3)}$ at that point are known.

Argument: Apply Newton's second law to a free body in the shape of a tetrahedron and let the height of the tetrahedron shrink to zero.

Consider the tetrahedron below. The point O is the origin and the apices are labeled A, B, and C.

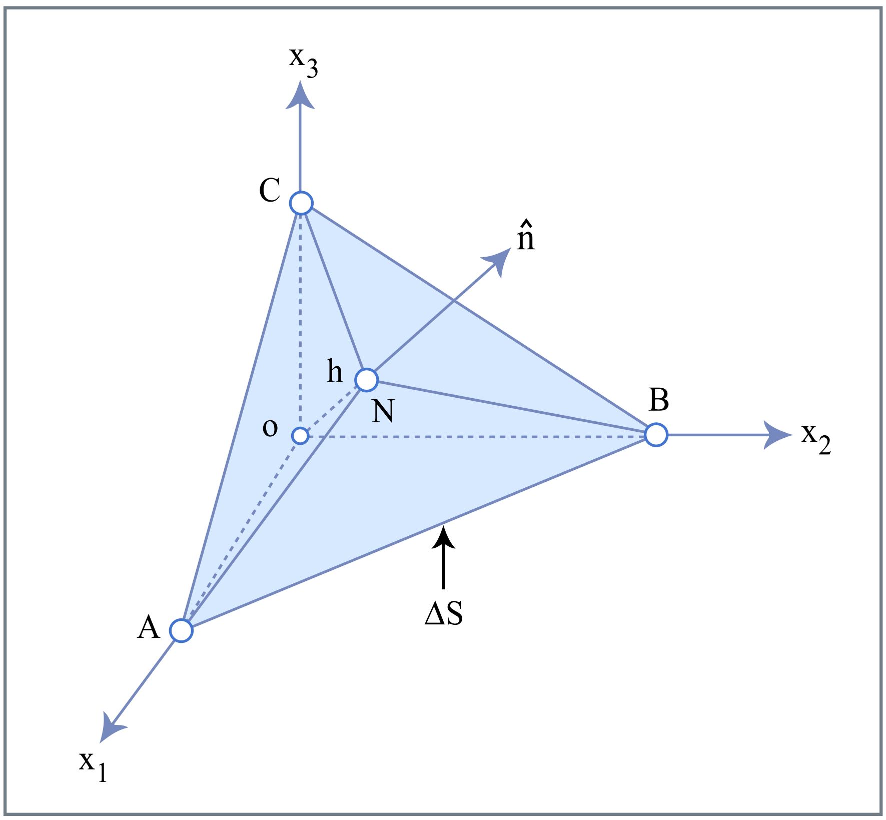

Figure 3.3

Figure by MIT OCW.

The relevant quantities are defined as follows:

ρ = density

$\underset{\sim}{F_i}$ = body force per unit mass in the *i* direction

$\underset{\sim}{a}_i$ = acceleration in the *i* direction

h = height of the tetrahedron, measured $\perp$ to ABC

ΔS = area of the oblique surface ABC

$\underset{\sim}{T}_i$ = the component of the traction vector on the oblique surface in the *i* direction

The mass of the tetrahedron is $\frac{1}{3}\rho h\Delta S$. The area of a face perpendicular to x_i is $n_i\Delta S$.

Force Balance on Tetrahedron

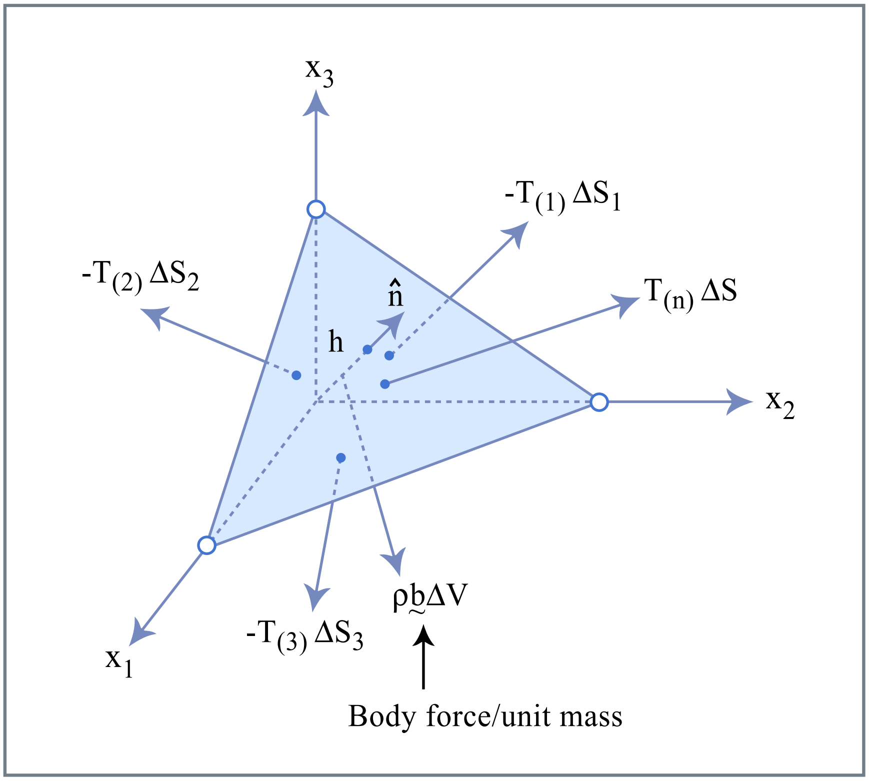

Figure 3.4

Figure by MIT OCW.

Consider the force balance in the *i*=1 direction. Overbars denote values averaged over a surface or volume.

$$\underset{\sim}{F}_1 = m\underset{\sim}{a}_1$$

$$\underset{\sim}{F}_1\left(\frac{1}{3}\bar{\rho}h\Delta S\right)+\bar{T}_{(n)}{}^1\Delta S-\bar{\sigma}_{11}\left(n_1\Delta S\right)-\bar{\sigma}_{21}\left(n_2\Delta S\right)-\bar{\sigma}_{31}\left(n_3\Delta S\right)=\left(\frac{1}{3}\bar{\rho}h\Delta S\right)\underset{\sim}{a}$$

Divide both sides by ΔS.

$$\underset{\sim}{F}_1\left(\frac{1}{3}\bar{\rho}h\right)+\bar{T}_{(n)}{}^1-\bar{\sigma}_{11}\left(n_1\right)-\bar{\sigma}_{21}\left(n_2\right)-\bar{\sigma}_{31}\left(n_3\right)=\left(\frac{1}{3}\bar{\rho}h\right)\underset{\sim}{a}$$

Allow h to approach zero in such a way that the surfaces and volume of the tetrahedron approach zero while the surfaces preserve their orientation. The body force and the mass both approach zero.

$$\bar{T}_{(n)}{}^1=\bar{\sigma}_{11}n_1+\bar{\sigma}_{21}n_2+\bar{\sigma}_{31}n_3$$

Performing the same force balance in the other two coordinate directions leads to expressions for the three traction components on an arbitrary plane.

$$\begin{aligned}T_{(n)_1}&=\sigma_{11}n_1+\sigma_{21}n_2+\sigma_{31}n_3\\T_{(n)_2}&=\sigma_{12}n_1+\sigma_{22}n_2+\sigma_{32}n_3\\T_{(n)_3}&=\sigma_{13}n_1+\sigma_{23}n_2+\sigma_{33}n_3\end{aligned}$$

The set of these three equations is called Cauchy's formula.

Different Notations

1. A general equation for the explicit expressions above is given by:

$$T_i=\sum_{j=1}^{3}\sigma_{ji}n_j$$

2. Summation notation is a way of writing summations without the summation sign Σ. To use it, simply drop the Σ and sum over repeated indices. The equation in summation notation is given by:

$$T_i = \sigma_{ji} n_j$$

3. The equation in matrix form is given by

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

For example, consider sliding block experiment. $\theta = 30^{\circ}$

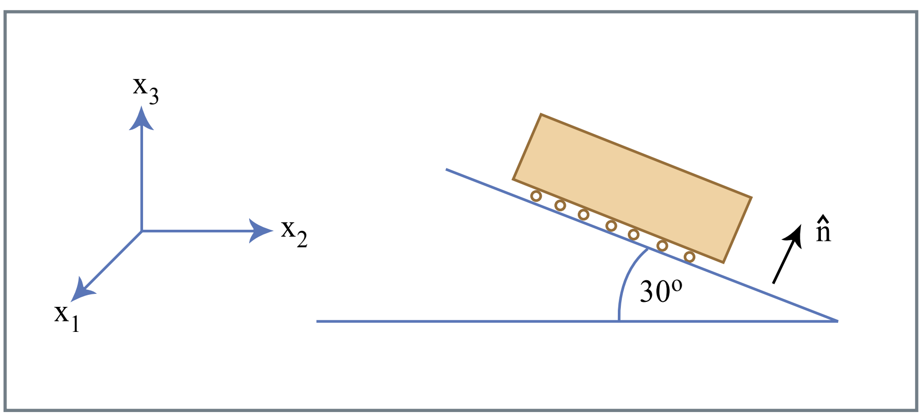

Figure 3.5

Figure by MIT OCW.

On the sliding plane: What is the traction $\underset{\sim}{T}$ in terms of σ_{ij}?

$$n = (0, \cos 60^{\circ}, \cos 30^{\circ})$$
$$= (0, 1/2, \sqrt{3}/2)$$

$$\underset{\sim}{T} = (0, 1/2, \sqrt{3}/2) \cdot \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

Features of the Stress Tensor

The stress tensor is a symmetric tensor, meaning that $\sigma_{ij} = \sigma_{ji}$. As a result, the entire tensor may be specified with only six numbers instead of nine.

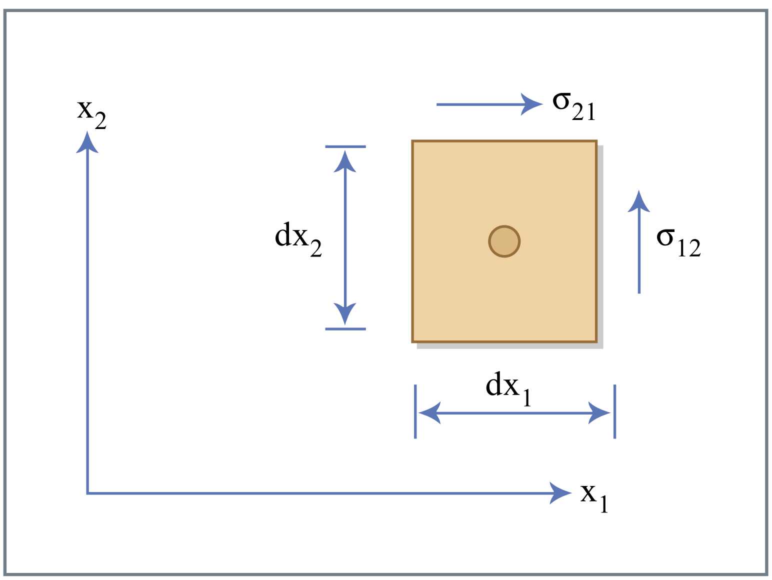

Figure 3.6

Figure by MIT OCW.

Consider the torques t acting on an element with sides dx_1 and dx_2.

$$t_3 = 2\sigma_{12}\frac{dx_1}{2}dx_2 - 2\sigma_{21}\frac{dx_2}{2}dx_1 = 0$$

$$\Rightarrow \sigma_{12} = \sigma_{21}$$

A similar argument shows $\sigma_{32} = \sigma_{23}$; $\sigma_{13} = \sigma_{31}$.

Shears are always “conjugate”.