Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, 2002

7.6 Transient longshore wind

[Ref]: Chapter 14, p. 195 ff, Cushman-Roisin
Csanady: Circulation in the Coastal Ocean

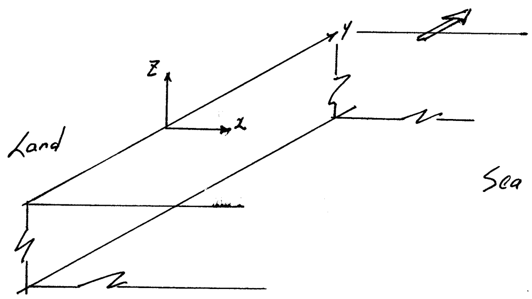

Figure 7.6.1: Longshore wind

In view of the last section, we ignore the bottom stress. Assume that the wind is uniform in space but transient in time, so that $\partial/\partial y = 0$, The flux equations are

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} = 0 \tag{7.6.1}$$

$$\frac{\partial U}{\partial t} - fV = -gh\,\frac{\partial \eta}{\partial x} \tag{7.6.2}$$

$$\frac{\partial v}{\partial t} + fU = \frac{\tau_y^S}{\rho}. \tag{7.6.3}$$

The boundary condition on the coast $x = 0: \quad U = 0.$

7.6.1 Sudden long-shore wind

Let the wind stress be

$$\tau_y^S = \begin{cases} 0, & t \le 0, \\ T, & t > 0. \end{cases} \tag{7.6.4}$$

the initial conditions are

$$\eta, U, V = 0, \quad t = 0, \quad \forall x. \tag{7.6.5}$$

This initial-boundary value problem can be solved by Laplace transform (Crépon, 1967). The solution consists of two parts: one part is oscillatory and decays with time; the other part increases monotonically with time. To avoid the complex mathematics we only examine the latter which is the dominant part for large time,

$$U = \bar{U}(x), \quad V = t\bar{V}(x), \quad \eta = t\bar{\eta}(x) \tag{7.6.6}$$

The oscillatory part is needed to ensure the initial condition on U.

It is easy to see from (7.6.1) to (7.6.3) that

$$\bar{\eta} + \frac{d\bar{U}}{dx} = 0 \tag{7.6.7}$$

$$f\bar{V} = gh\frac{d\bar{\eta}}{dx} \tag{7.6.8}$$

$$\bar{V} + f\bar{U} = T/\rho \tag{7.6.9}$$

These three equations can be combined into one :

$$\frac{d^2\bar{U}}{dx^2} - \frac{f^2}{gh}\bar{U} = -\frac{fT}{\rho gh} \tag{7.6.10}$$

The solution satisfies no flux on the coast is

$$\bar{U} = \frac{T}{\rho f}\left(1 - e^{-x/R_o}\right) \tag{7.6.11}$$

where

$$R_o = \frac{\sqrt{gh}}{f} \tag{7.6.12}$$

is called the Rossby radius of deformation. Since $f = 10^{-4}$ 1/s, in a shallow sea of $h = 10$ m the Rossby radius is about 10^5 m = 100 km.

It is easy to find that

$$\eta = t\bar{\eta} = -t\frac{T}{\rho gh}e^{-x/R_o} \tag{7.6.13}$$

and

$$V = t\bar{V} = t\frac{T}{\rho f}e^{-x/R_o} \tag{7.6.14}$$

Clearly when $x/R_o \gg 1$, the coast line has no influence. The flux is $U = T/\rho f, V = 0$, and is inclined to the right of the wind by 90 degrees, as predicted by the Ekman layer theory. The sea surface sinks near the coast if $T > 0$ (coast is on the left of wind) and rises if $T < 0$ (coast is on the right of wind).

7.6.2 Sinusoidal wind stress

We now consider

$$\tau_y^S = \Re\left(\tau_0 e^{-i\omega t}\right) = \tau_o \sin\omega t \qquad (7.6.15)$$

Let

$$(\eta, U, V) = \Re\left[(\eta_0, U_0, V_0)\, e^{-i\omega t}\right] \qquad (7.6.16)$$

The symbol $\Re$ (real part of) will be omitted for brevity.

Let us calculate the total flux (The boundary layers can be studied later.),

$$-i\omega\eta_0 + \frac{dU_0}{dx} = 0 \qquad (7.6.17)$$

$$-i\omega U_0 - fV_0 = -gH\,\frac{d\eta_0}{dx} \qquad (7.6.18)$$

$$-i\omega V_0 + fU_0 = i\frac{\tau_0}{\rho}. \qquad (7.6.19)$$

An equation for a single variable can be obtained. For example by solving Eqns. (7.6.18) and (7.6.19) for U_0 and V_0, we get

$$\begin{aligned}
U_0 &= \frac{\begin{vmatrix} -gH\,\frac{d\eta_0}{dx} & -f \\ i\tau_0 & -i\omega \end{vmatrix}}{\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix}} \\
&= \frac{i\omega gh\,\frac{d\eta_0}{dx} + i\tau_0 f}{-\omega^2 + f^2} \\
U_0 &= \frac{i\omega gh}{f^2 - \omega^2}\left(\frac{d\eta_0}{dx} + \frac{\tau_0 f}{\rho\omega gh}\right) \qquad (7.6.20)
\end{aligned}$$

Differentiate Eqn. (7.6.20) and use Eqn. (7.6.17)

$$-i\omega\eta_0 + \frac{i\omega gh}{f^2 - \omega^2}\frac{d^2\eta_0}{dx^2} = 0$$

or

$$\frac{d^2\eta_0}{dx^2} - \frac{f^2 - \omega^2}{gh}\,\eta_0 = 0 \qquad (7.6.21)$$

We now distinguish two cases.

Low frequency: $\omega < f$

The solution to (7.6.21) bounded at infinity is

$$\eta_0 = A\, e^{-x/R_0} \tag{7.6.22}$$

where

$$R_0 = \sqrt{\frac{gh}{f^2 - \omega^2}}. \tag{7.6.23}$$

is the modified Rossby radius.

Applying the B.C. on the coast: $U_0 = 0$, we get from (7.6.20),

$$\frac{d\eta_0}{dx} = \frac{-f}{\rho g H}\,\frac{\tau_0}{\omega} \overset{(7.6.22)}{=} \frac{-A}{R_0}.$$

and,

$$A = \frac{\tau_0}{\omega}\,\frac{f}{\rho g H}\,R_0$$

Hence

$$\eta_0 = \frac{f\tau_0}{\rho\omega g H}\,R_0\, e^{-x/R_o}$$

and finally

$$\eta = \frac{f\tau_0}{\rho\omega g H}\,R_0\, e^{-x/R_0}\, e^{-i\omega t} \tag{7.6.24}$$

Now

$$\eta_t = -\frac{if\tau_0}{\rho g H}\,R_0\, e^{-x/R_o}\, e^{-i\omega t} = -U_x.$$

from Eqn. (7.6.1). Integrating with respect to x,

$$U = \frac{if\tau_o R_0^2}{\rho g h}\left(1 - e^{-x/R_0}\right)\, e^{-i\omega t}. \tag{7.6.25}$$

From Eqn. (7.6.20)

$$\begin{aligned}
-i\omega V_0 &= -fU_0 + i\tau_0/\rho \\
&= \frac{-if^2\tau_0 R_0^2}{\rho g H}\left(1 - e^{-x/R_0}\right) \quad + \frac{i\tau_0}{\rho} \\
&= \frac{i\tau_0}{\rho}\left[1 - \frac{f^2}{gh}\,R_0^2\left(1 - e^{-x/R_0}\right)\right]
\end{aligned}$$

$$V_0 \;=\; -\frac{\tau_0}{\rho\omega}\left[1 - \frac{f^2}{f^2 - \omega^2}\left(1 - e^{-x/R_0}\right)\right]$$

$$
\begin{aligned}
&= -\frac{\tau_0}{\rho\omega}\,\frac{f^2}{f^2-\omega^2}\left[\frac{f^2-\omega^2}{f^2}-1+e^{-x/R_0}\right]\\
&= \frac{\tau_0}{\rho\omega}\,\frac{\omega^2}{f^2}\,\frac{f^2}{f^2-\omega^2}\left[1-\frac{f^2}{\omega^2}\,e^{-x/R_0}\right]\\
&= \frac{\tau_0\omega/\rho_0}{f^2-\omega^2}\left[1-\frac{f^2}{\omega^2}\,e^{-x/R_0}\right].
\end{aligned}
$$

Let us summarize the results in real form,

$$
\tau_y^S = \tau_0 \sin\omega t \tag{7.6.26}
$$

$$
\eta = \frac{f\tau_0}{\rho\omega g h}\,R_0\,e^{-x/R_0}\,\cos\,\omega t \tag{7.6.27}
$$

$$
U = \frac{f\tau_o R_0^2}{\rho g h}\left(1-e^{-x/R_0}\right)\,\sin\,\omega t \tag{7.6.28}
$$

$$
V = \frac{\tau_0\omega}{\rho\,(f^2-\omega^2)}\left(1-\frac{f^2}{\omega^2}\,e^{-x/R_0}\right)\,\cos\,\omega t. \tag{7.6.29}
$$

If $\tau_0 < 0$ (or > 0), i.e., the coast is on the right (left) of wind, the sea level near the coast rises (sinks).

High frequency : $\omega > f$

Of the two possible oscillatory solutions to (7.6.21), we must choose the one that represents outgoing waves at infinty (the radiation condition),

$$
\eta_0 = Ae^{ikx}, \tag{7.6.30}
$$

where the wavenumber is the inverse of the modified Rossby radius of deformation,

$$
k = \sqrt{\frac{\omega^2-f^2}{gh}} \tag{7.6.31}
$$

We leave it to the reader to show that, in complex form,

$$
\eta = \frac{i\tau_0 f}{\rho g h\omega k}e^{ikx-i\omega t} \tag{7.6.32}
$$

$$
U = -\frac{i\tau_0 f}{\rho(\omega^2-f^2)}\left(1-e^{ikx}\right)e^{-i\omega t} \tag{7.6.33}
$$

$$
V = \;= -\frac{\tau_0}{\rho\omega}\left[1+\frac{f^2}{\omega^2-f^2}\left(1-e^{ikx}\right)\right]e^{-i\omega t} \tag{7.6.34}
$$

or, in real form,

$$
\eta = -\frac{\tau_0 f}{\rho g h\omega k}\sin(kx-\omega t), \tag{7.6.35}
$$

$$U = -\frac{\tau_0 f}{\rho(\omega^2 - f^2)} \left(\sin \omega t + \sin(kx - \omega t)\right) \tag{7.6.36}$$

$$V = \; = -\frac{\tau_0}{\rho\omega} \left[1 + \frac{f^2}{\omega^2 - f^2} \left(\cos \omega t - \cos(kx - \omega t)\right)\right] \tag{7.6.37}$$