Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, 2002

Chapter 7. Geophysical Fluid Dynamics of Coastal Region

[Ref]: Pedlosky *Geophysical Fluid Dynamics* Springer-Verlag
Csanady: Circulation in the Coastal Ocean, Kluwer

7.1 Equations of Motion in Rotating Coordinates

Since the earth is rotating about the polar axis, the coordinate system fixed on earth is rotating. We need to know how to express the time rate of change of dynamical quantities in the rotating coordinates.

A vector fixed in the rotating coordinate system is rotating in the fixed (inertial) coordinate system. Consider therefore a vector rotating in the inertial frame of reference.

7.1.1 Vector of constant magnitude

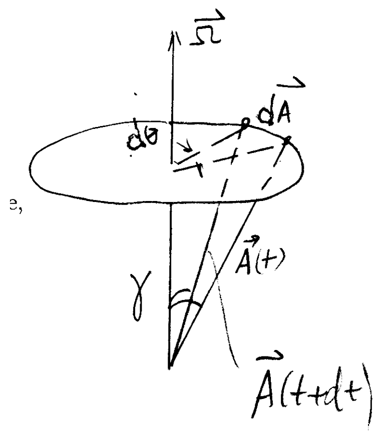

Figure 7.1.1: Vector $\vec{A}(t)$ rotating at the angular velocity $\vec{\Omega}$.

If $\vec{A} = A_i \vec{e}_i$ has a constant magnitude but is rotating about an axis at the angular vecloty $\vec{\Omega}$, what is the rate of change $d\vec{A}/dt$ in the fixed coordinate (inertial) system? Let

$$d\vec{A} = \vec{A}(t+dt) - \vec{A}(t)$$

From Figure 7.1.1,

$$\left(\frac{d\vec{A}}{dt}\right)_I = \vec{e}\,|A| \sin\gamma \,\frac{d\theta}{dt},$$

where subscript I signifies "inertial system" and $\vec{e}$ is the unit-vector along $d\vec{A}$. Note $\vec{e} \perp \vec{A}$ and $\vec{e} \perp \vec{\Omega}$. Hence,

$$\vec{e} = \frac{\vec{\Omega} \times \vec{A}}{\mid \vec{\Omega} \times \vec{A} \mid}.$$

and ,

$$\left(\frac{d\vec{A}}{dt}\right)_I = \frac{\vec{\Omega} \times \vec{A}}{\mid \vec{\Omega} \times \vec{A} \mid} \mid \vec{A} \mid \sin\gamma \frac{d\theta}{dt},$$

Since

$$\frac{d\theta}{dt} = \Omega,$$

$$\vec{\Omega} \times \vec{A} \mid = \Omega \mid \vec{A} \mid \sin\gamma.$$

it follows that

$$\left(\frac{d\vec{A}}{dt}\right)_I = \vec{\Omega} \times \vec{A}. \tag{7.1.1}$$

In particular, let $\vec{A} = \vec{e}_i, i = 1, 2, 3$ be a base vector in the rotating frame of reference,

$$\vec{A} = \vec{e}_i \neq \vec{e}$$

Then

$$\left.\frac{d\vec{e}_i}{dt}\right|_I = \vec{\Omega} \times \vec{e}_i. \tag{7.1.2}$$

7.1.2 A vector of variable magnitude

Let

$$\vec{B} = B_i \, \vec{e}_i$$

be any non-constant vector in the rotating frame, and let

$$\left(\frac{d\vec{B}}{dt}\right)_R = \frac{dB_i}{dt} \vec{e}_i$$

denote its rate of change in the rotating frame. then

$$\left(\frac{d\vec{B}}{dt}\right)_I = \frac{dB_i}{dt} \vec{e}_i + B_i \frac{d\vec{e}_i}{dt} = \left(\frac{d\vec{B}}{dt}\right)_R + B_i \vec{\Omega} \times \vec{e}_i = \left(\frac{d\vec{B}}{dt}\right)_R + \vec{\Omega} \times \vec{B}. \tag{7.1.3}$$

In particular, if $\vec{B} = \vec{r}$ is the position of a fluid particle

$$\left.\frac{d\vec{r}}{dt}\right|_I = \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\Omega} \times \vec{r},$$

Note that $\vec{r}$ is the same in any coordinate system. Now $(d\vec{r}/dt)_I$ is the velocity seen in the inertial frame of reference and $(d\vec{r}/dt)_R$ is the velocity seen in the rotating frame of reference, i.e.,

$$\vec{q}_I = \vec{q}_R + \vec{\Omega} \times \vec{r}; \tag{7.1.4}$$

Next we let $\vec{q}_R$ be the velocity vector of fluid in the rotating frame of reference; its rates of change in the two frames of reference are related by

$$\left.\frac{d\vec{q}_R}{dt}\right|_I = \left.\frac{d\vec{q}_R}{dt}\right|_R + \vec{\Omega} \times \vec{q}_R. \tag{7.1.5}$$

Taking the time derivative of (7.1.4), and assuming that the angular acceleration of earth to be zero,

$$\frac{d\vec{\Omega}}{dt} = 0$$

we get

$$\begin{aligned}
\left(\frac{d\vec{q}_I}{dt}\right)_I &= \left(\frac{d\vec{q}_R}{dt}\right)_I + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_I \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \vec{\Omega} \times \vec{q}_R + \vec{\Omega} \times \left[\left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r}\right] \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \underbrace{2\vec{\Omega} \times \vec{q}_R}_{\text{Coriolis acc.}} + \underbrace{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centripetal}}
\end{aligned} \tag{7.1.6}$$

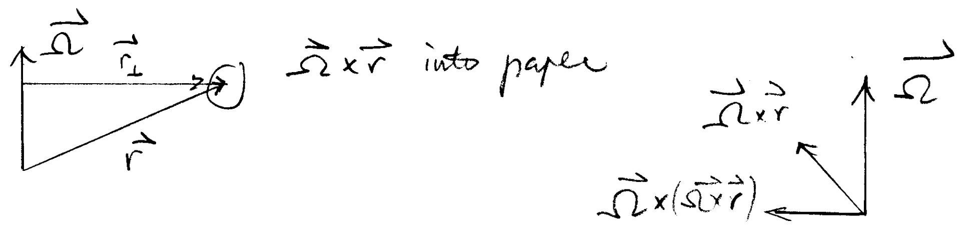

Figure 7.1.2: Coriolis force, position vector and angular velocity

The second term on the right is the Coriolis force, being perpendicular to both $\vec{q}$ and $\vec{\Omega}$. The last term represents the centripetal force

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -|\Omega|^2\, \vec{r_\perp},$$

See Figure 7.1.2 for the geometric relations.

The centripetal force may be written in terms of a centripetal force potential ϕ_c where

$$\phi_c = \frac{1}{2}\,(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = \frac{1}{2}|\Omega|^2 r_\perp^2. \tag{7.1.7}$$

so that

$$-\nabla\phi_c = -\frac{d\phi_c}{dr_\perp}\vec{e}_\perp = -|\Omega|^2\, \vec{r_\perp}, \tag{7.1.8}$$

7.1.3 Summary of governing equations in rotating frame of reference:

Continuity:

$$\nabla \cdot \vec{q} = 0 \tag{7.1.9}$$

In the coordinate system rotating at the constant angular velocity, the momentum equation reads, after dropping subscripts R

$$\rho\left(\frac{d\vec{q}}{dt} + 2\vec{\Omega}\times\vec{q}\right) = -\nabla p + \rho\nabla(\phi_g + \phi_c) + \mu\nabla^2\vec{q} \tag{7.1.10}$$

where

$$\phi_g = gz \qquad \phi_c = \frac{1}{2}\left(\vec{\Omega}\times\vec{r}\right)\cdot\left(\vec{\Omega}\times\vec{r}\right)$$

7.1.4 Dimensionless parameters

$$\begin{aligned}
\frac{\frac{\partial\vec{q}}{\partial t}}{2\Omega\times\vec{q}} &= \frac{O\left(\frac{U}{T}\right)}{\Omega U} = O\left(\frac{1}{\Omega T}\right)\\
\frac{\vec{q}\cdot\nabla\vec{q}}{2\vec{\Omega}\times\vec{q}} &= \frac{U^2/L}{2\Omega U} = \frac{U}{2\Omega L} = \text{Rossby number}\\
\frac{\nu\nabla^2\vec{q}}{2\Omega\times\vec{q}} &= \frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2} = \text{Ekman number}\\
\nabla\phi_g &= \vec{g}\\
\nabla\phi_c &= \Omega^2\vec{r}_L
\end{aligned}$$

For numerical estimate, we take $\Omega = \frac{1}{12\text{ hrs}} = 2.31\times10^{-5}s^{-1}$ and $r =$ earth radius $= 6400$ km. Then $\omega^2 r \sim \left(2.31\times10^{-5}\right)^2 \times 6.4\times10^6 \sim 3\times10^{-3}\text{m/s}^2$ while $g \sim 10\text{m/s}^2$. Hence $g \gg \Omega^2 r$; gravity is more important than centripetal force.

7.1.5 Coriolis force

Refering to the right of Figure 7.1.3

$$\vec{\Omega} = \vec{i}\,(-\Omega\cos\theta) + \vec{j}(0) + \vec{k}(\Omega\sin\theta)$$

Introducing the spherical polar coordinates as in the left of Figure Refering to the right of Figure 7.1.3, with θ being the latitude. The Coriolis force is

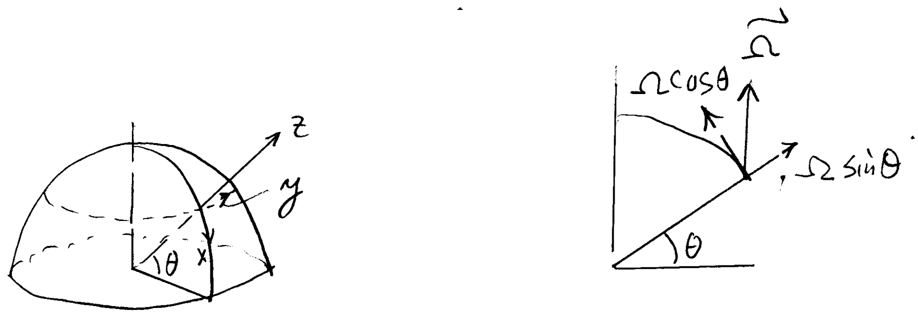

Figure 7.1.3: The Northern hemisphere.

$$2\vec{\Omega} \times \vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\Omega\cos\theta & o & 2\Omega\sin\theta \\ u & v & w \end{vmatrix}$$

$$= \quad \vec{i}\,(-2\Omega v \sin\theta) + \vec{j}\,(2\Omega u \sin\theta + 2\Omega w \cos\theta) + \vec{k}\,(-2\Omega v \cos\theta)$$

Consider shallow waters where the depth D is much less than the horizontal length L, i.e., $D \ll L$, and compare the two terms in the y direction of $(\vec{j})$

$$\frac{2\Omega w \cos\theta}{2\Omega u \sin\theta} = \frac{w}{u}\cot\theta = O\left(\frac{D}{L}\right)\cot\theta \ll 1$$

except along the equator where $\theta = 0$

In the z direction of $(\vec{k})$,

$$\frac{-2\Omega v\cos\theta}{\frac{1}{\rho}\frac{\partial_d p}{\partial z}} = \frac{-2\Omega u\cos\theta}{\frac{L}{D}\frac{\partial p_d}{\partial x}} = \frac{D}{L}\frac{-2\Omega u\cos\theta}{\max\left(\frac{U}{T},\frac{U^2}{L},\Omega U\right)} = O\left(\frac{D}{L}\right) \ll 1$$

Hence in shallow seas

$$2\vec{\Omega} \times \vec{q} \cong \vec{i}(-2\Omega v\sin\theta) + \vec{j}(2\Omega u \sin\theta)$$

Define

$$f = 2\Omega\sin\theta \tag{7.1.11}$$

to be the Coriolis parameter, then

$$2\vec{\Omega} \times \vec{q} = -fv\,\vec{i} + fu\,\vec{j} \tag{7.1.12}$$

In the northern hemisphere, $0 < \theta < \pi/2$.