Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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2-2lubri.tex

2.2 Lubrication approximation for flow in a thin layer

An essential first step of any analytical approximtation is the art of scaling, which we shall emphasize repeatedly throughout this course.

Let H be the characteristic depth and L the characteristic length in the direction of the flow, and assume a shallow layer, i.e.,

$$H/L \ll 1 \tag{2.2.1}$$

Let U be the scale of u, then by continuity, the scale of v must be $U\frac{H}{L}$ in order not to violate mass conservation. Leaving the velocity and pressure scales U, P undermined for the time being, we introduce the following scales and normalized variables, denoted by primes,

$$t = Tt',\ x = Lx', y = Hy',\ u = Uu',\ v = U\frac{H}{L}v', \quad p = Pp', \tag{2.2.2}$$

The normalized continuity equation is

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{2.2.3}$$

Both terms are equally important, reflecting the holyness of the law of mass concentration. The longitudinal momentum equation is normalized to

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) = g\sin\theta - \frac{P}{\rho L}\frac{\partial p'}{\partial x'} + \frac{\nu U}{H^2}\left(\frac{H^2}{L^2}\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) \tag{2.2.4}$$

Dividing by $\frac{\nu U}{H^2}$, we get

$$\frac{H^2}{\nu T}\frac{\partial u}{\partial t} + \frac{UH}{\nu}\frac{H}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) = \frac{g\sin\theta H^2}{\nu U} - \frac{PH^2}{\rho L\nu U}\frac{\partial p'}{\partial x'} + \left(\frac{H^2}{L^2}\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) \tag{2.2.5}$$

For a shallow layer ($H/L \ll 1$) we assume in addition,

$$\frac{UH}{\nu} = O(1) \tag{2.2.6}$$

and

$$\frac{H^2}{\nu T} \ll 1 \tag{2.2.7}$$

Omitting terms of the order H/L and smaller, the above equation can be approximated to the leading order by

$$0 = \frac{g \sin\theta H^2}{\nu U} - \frac{P H^2}{\rho L \nu U} \frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2} \tag{2.2.8}$$

or in dimensional form,

$$0 = g \sin\theta - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{2.2.9}$$

All inertia terms are inconsequential; the most important balance is among gravity, the pressure gredient and the dominant viscous stress. This balance also implies a pressure scale,

$$P = \frac{\rho L \nu U}{H^2} \tag{2.2.10}$$

From the transverse momentum equation,

$$\frac{H}{L}\left[\frac{U}{T}\frac{\partial v'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right)\right] = -g\cos\theta - \frac{P}{\rho H}\frac{\partial p'}{\partial y'} + \frac{H}{L}\frac{\nu U}{H^2}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) \tag{2.2.11}$$

or

$$\begin{aligned} &\frac{H^2}{L^2}\left\{\frac{H^2}{\nu T}\frac{\partial v'}{\partial t'} + \frac{UH}{\nu}\frac{H}{L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right)\right\} = \\ &\quad -\frac{g\sin\theta H^2}{\nu U}\frac{H}{L\tan\theta} - \frac{PH^2}{\rho L \nu U}\frac{\partial p'}{\partial y'} + \frac{H^2}{L^2}\left(\frac{H^2}{L^2}\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) \end{aligned} \tag{2.2.12}$$

Either for finite bed slope or for small slope but

$$O\left(\frac{H}{L}\right) = \tan\theta \ll 1 \tag{2.2.13}$$

the left hand side above is negligible with an error of $O(H/L)^3$. In physical variables the approximate result is

$$0 = -g\cos\theta - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.2.14}$$

Not only the inertia terms are insignificant, the pressure is hydrostatic. This balance also implies the pressure scale

$$P = \rho g H \cos\theta \tag{2.2.15}$$

Note that (2.2.10) and (2.2.15) together implies the velocity scale

$$U = \frac{H}{L}\frac{gH\cos\theta}{\nu} \tag{2.2.16}$$

The distinguishing feature of negligible inertia is shared by the slow flow through thin gaps of bearings in the theory of lubrication. Hence (2.2.9) and (2.2.14) can be called the lubrication approximation.

We leave it as an exercise to show by similar normalization, that the dynamic boundary conditions on $y = h$ can be approximated to the leading order by

$$\frac{\partial u}{\partial y} = 0 \tag{2.2.17}$$

for the tangential stress, and

$$p = 0 \tag{2.2.18}$$

for the normal stress. It follows by integrating (2.2.14)) that

$$p(x, y, t) = \rho g \cos\theta [h(x, t) - y] \tag{2.2.19}$$

The longitudinal momentum equation can also be readily integrated,

$$u = -\frac{\rho g}{\mu}\left(\sin\theta - \cos\theta \frac{\partial h}{\partial x}\right)\left(\frac{y^2}{2} - hy\right) \tag{2.2.20}$$

The total discharge is

$$Q = \overline{u}h = \int_0^h u\, dy = \frac{\rho g h^3}{3\mu}\left(\sin\theta - \cos\theta \frac{\partial h}{\partial x}\right) \tag{2.2.21}$$

which can be inserted in (**??**) to give

$$\boxed{\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left[h^3\left(\sin\theta - \cos\theta \frac{\partial h}{\partial x}\right)\right] = 0} \tag{2.2.22}$$

This is a nonlinear diffusion equation governing the evolution of the fluid depth.

In the special limit of a uniform flow, $\partial/\partial x \equiv 0$. The velocity profile is then

$$u = \frac{\rho g h^2}{\mu}\sin\theta\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) \tag{2.2.23}$$

with h being a pure constant. The coresponding discharge is

$$Q = \frac{\rho g h^3}{3\mu}\sin\theta \tag{2.2.24}$$