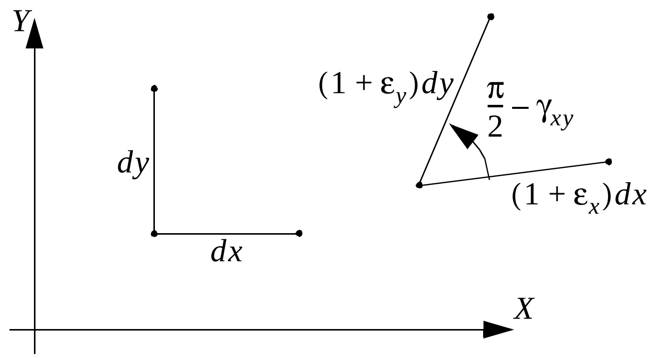
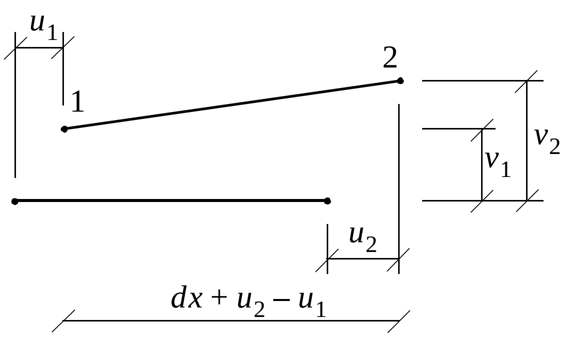
5.1 Deformation Analysis



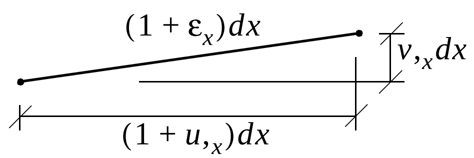
5.1.1 Deformation of a fiber dx initially aligned with x.



$$u_2 - u_1 = \Delta u$$

$$dx + u_2 - u_1 = dx + du$$

$$dx\left(1 + \frac{du}{dx}\right) = dx(1 + u,_x)$$



$$v_2 - v_1 = \Delta v$$

$$dv = \frac{dv}{dx}dx = v,_x dx$$

$$(1 + \varepsilon_x)^2 = (1 + u,_x)^2 + v,_x^2$$

$$1 + 2\varepsilon_x + \varepsilon_x^2 = 1 + 2u,_x + u,_x^2 + v,_x^2$$

$$\varepsilon_x(2 + \varepsilon_x) = u,_x(2 + u,_x) + v,_x^2$$

$$\varepsilon_x\left(1 + \frac{\varepsilon_x}{2}\right) = u,_x\left(1 + \frac{u,_x}{2}\right) + \frac{v,_x^2}{2}$$

For small strain ($\varepsilon_x \ll 1$)

$$\varepsilon_x = u,_x\left(1 + \frac{u,_x}{2}\right) + \frac{v,_x^2}{2}$$

If the member does not experience large relative rotations, then the non-linear $u,_x$ term can be ignored.

Then for small relative rotations

$$\varepsilon_x = u,_x + \frac{1}{2} v,_x^2$$

5.1.2 Deformation of a fiber dy initially aligned with y.

Proceed as done for dx to obtain:

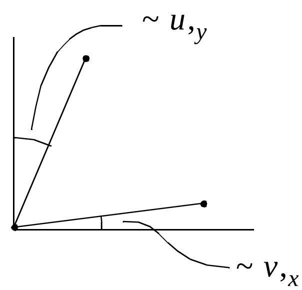
For small strain

$$\varepsilon_y = v,_y\left(1 + \frac{v,_y}{2}\right) + \frac{1}{2} u,_y^2$$

For small relative rotations

$$\varepsilon_y = v,_y + \frac{1}{2} u,_y^2$$

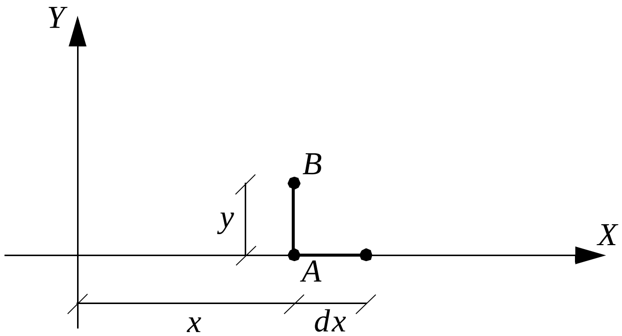
5.1.3 Shear deformation



For small rotations

$$\gamma \cong u,_y + v,_x$$

5.2 Straight Beams - Non-Linear Analysis for Plane Members



5.2.1 Axial Strain

$$u_B = u_A - y\sin\beta$$

$$v_B = v_A - y(\cos\beta - 1) \cong v_A$$

For small rotations

$$\beta^2 \ll 1$$

$$\sin\beta \cong \beta$$

$$\cos\beta \cong 1$$

$$\varepsilon_x\big|_B = \frac{\partial(u_B)}{\partial x} + \frac{1}{2}\left(\frac{\partial(v_B)}{\partial x}\right)^2$$

$$\varepsilon_x\big|_B = \frac{\partial}{\partial x}(u_A - y\sin\beta) + \frac{1}{2}\left(\frac{\partial v_A}{\partial x}\right)^2$$

$$\varepsilon_x\big|_B = u_{A},_x - y\beta,_x + \frac{1}{2}(v_{A},_x)^2$$

$$\varepsilon_x\big|_B = \varepsilon_x\big|_A - y\beta,_x = \varepsilon_{xo} - y\chi$$

5.2.2 Shear Strain

$$\gamma_B = \frac{\partial u_B}{\partial y} + \frac{\partial v_B}{\partial x} = \frac{\partial}{\partial y}(u_A - y\sin\beta) + \frac{\partial}{\partial x}(v_A)$$

$$\gamma_B = -\beta + v_{A},_x$$

5.2.3 Summarizing

Define $\varepsilon_x(y) = \varepsilon_x'$

$\gamma = \gamma'$

$$\varepsilon_x' = \varepsilon_x - y\beta,_x$$

$$\varepsilon_x = u,_x + \frac{1}{2}v,_x^2$$

$$\gamma' = v,_x - \beta$$

5.2.4 Apply the principle of virtual displacements

- Take a deformable body resisting some external loads
- Produce a perturbation (a virtual displacement)
- If body is in state of equilibrium, the first order work done by the stresses is equal to the first order work done by the externally applied forces

i.e. $\int(\sigma \cdot \partial\varepsilon)dVol = \sum \text{Forces} \cdot \text{Virtual Displacements}$

For a beam: Internal virtual work is caused by stresses σ_x, τ_{xy}

$$\int_x\int_A \{(\sigma_x'\delta\varepsilon_x' + \tau_{xy}'\delta\gamma_{xy})dA\}dx = \sum \text{Forces} \cdot \text{Virtual Displacements}$$

$$\varepsilon_x' = \varepsilon_x - y\beta,_x = \varepsilon_x - y\chi$$

$$\varepsilon_x = u,_x + \frac{1}{2}v,_x^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$(\varepsilon_x + \delta\varepsilon_x) = \frac{\partial(u + \delta u)}{\partial x} + \frac{1}{2}\left(\frac{\partial(v + \delta v)}{\partial x}\right)^2$$

$$(\varepsilon_x + \delta\varepsilon_x) = \frac{\partial u}{\partial x} + \frac{\partial \delta u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial \delta v}{\partial x}\right)^2$$

$$(\varepsilon_x + \delta\varepsilon_x) = \frac{\partial u}{\partial x} + \frac{\partial \delta u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial \delta v}{\partial x} + \frac{1}{2}\left(\frac{\partial \delta v}{\partial x}\right)^2$$

Then $$\delta\varepsilon_x = \frac{\partial \delta u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial \delta v}{\partial x} + \frac{1}{2}\left(\frac{\partial \delta v}{\partial x}\right)^2 = \delta u,_x + v,_x \delta v,_x$$

$$y(\chi + \delta\chi) = y\frac{\partial(\beta + \delta\beta)}{\partial x} = y\frac{\partial \beta}{\partial x} + y\frac{\partial \delta\beta}{\partial x}$$

$$y\delta\chi = y\frac{\partial \delta\beta}{\partial x} = y\partial\beta,_x$$

Therefore

$$\delta\varepsilon_x' = \delta\varepsilon_x - y\delta\beta,_x$$

$$\gamma' = v,_x - \beta = \frac{\partial v}{\partial x} - \beta = \gamma$$

$$\gamma' + \delta\gamma' = \frac{\partial(v + \delta v)}{\partial x} - \beta - \delta\beta = \gamma + \delta\gamma$$

$$\delta\gamma' = \frac{\partial \delta v}{\partial x} - \delta\beta = \delta v,_x - \delta\beta = \delta\gamma$$

Right hand side of the virtual work equation

$$\int_x \int_A \{(\sigma_x'\delta\varepsilon_x' + \tau_{xy}'\delta\gamma_{xy})dA\}dx$$

$$\int_x \int_A \sigma_x'\delta\varepsilon_x' dAdx = \int_x \int_A (\sigma_x'\delta\varepsilon_x - \sigma_x y\delta\beta,_x)dAdx$$

$$\int_x \int_A \sigma_x'\delta\varepsilon_x' dAdx = \int_x (F\delta\varepsilon_x + M\delta\beta,_x)dx$$

$$\int_x \int_A \tau_{xy}'\delta\gamma'_{xy} dAdx = \int_x V\delta\gamma dx$$

Giving

$$RHS = \int_x (F\delta\varepsilon_x + M\delta\beta,_x + V\delta\gamma)dx$$

Integrating by parts $\int uv' = uv - \int u'v$

$$RHS = (F\delta u + Fv,_x\delta v + M\delta\beta + V\delta v)\Big|_0^L - \int_x \left(\delta uF,_x + \delta v\left(\frac{\partial(Fv,_x)}{\partial x} + V,_x\right) + \delta\beta(M,_x + V)\right)dx$$

Left hand side of the virtual work equation

$$LHS = \int_x b_y\delta v dx + \int_x b_x\delta u dx + \int_x m\delta\beta dx + \sum P \cdot \delta p$$

Differentiating both RHS and LHS wrt x, recognizing that the first RHS term is a constant and that both the second RHS term and the LHS are integrated over the same interval, we get

$$(F,_x + b_x)\delta u + (V,_x + (Fv,_x),_x + b_y)\delta v + (M,_x + V + m)\delta\beta = 0$$

Since δu, δv, and $\delta\beta$ are independent, each term becomes and independent expression

$$F,_x + b_x = 0$$

$$V,_x + (Fv,_x),_x + b_y = 0$$

$$M,_x + V + m = 0$$

These are the governing equations of equilibrium for a non-linear member.

Note: in $V,_x + (Fv,_x),_x + b_y = 0$, $(Fv,_x),_x$ = source of P-δ effect.

5.3 Compatibility Equations

$$F = \int_A \sigma_x' dA = \int_A E\varepsilon_x' dA = \int_A E\left(u,_x + \frac{1}{2}v,_x^2 - y\beta,_x\right) dA$$

$$F = u,_x \int_A EdA + \frac{1}{2}v,_x^2 \int_A EdA - \beta,_x \int_A yEdA$$

If y is measured from the mechanical centroid $\int_A yEdA = 0$

Then

$$F = D_S\left(u,_x + \frac{1}{2}v,_x^2\right)$$

where $D_S = \int_A EdA$

$$M = -\int_A y\sigma_x' dA = -\left(u,_x + \frac{1}{2}v,_x^2\right)\int_A EydA + \beta,_x \int_A Ey^2 dA$$

$$M = D_B \beta,_x$$

where $D_B = \int_A Ey^2 dA$

$$M = \int_A \tau_{xy} dA = \int_A G\gamma' dA = \int_A G(v,_x - \beta) dA$$

$$V = D_T(v,_x - \beta)$$

$$D_T = \int_A GdA$$

Summarizing

$$F,_x + b_x = 0$$

$$V,_x + (Fv,_x),_x + b_y = 0$$

$$M,_x + V + m = 0$$

$$F = D_S\left(u,_x + \frac{1}{2}v,_x^2\right)$$

$$M = D_B\beta,_x$$

$$V = D_T(v,_x - \beta)$$

Boundary conditions

u or F_x

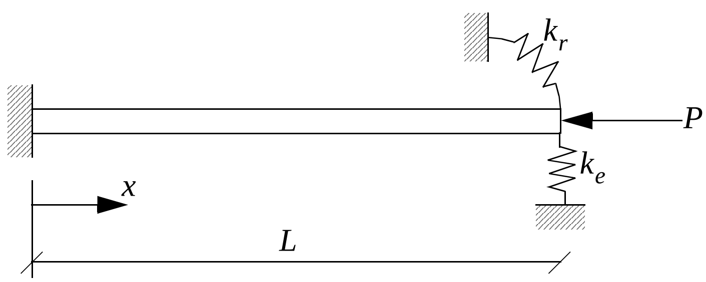
β or M

v or $V + Fv,_x = P_y$

Note P_y = effective shear

5.3.1 Examples of boundary conditions

a



@ $x = 0$

$u = v = \beta = 0$

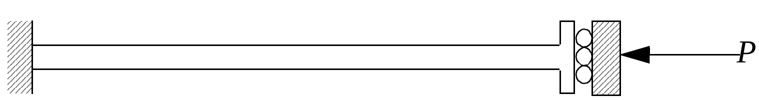
@ $x = L$

$F = -P$

$M = -k_r\beta(L)$; $M = 0$ without spring

$V - v,_x P = -k_e v(L)$

b



@ $x = 0$

$u = v = \beta = 0$

@ $x = L$

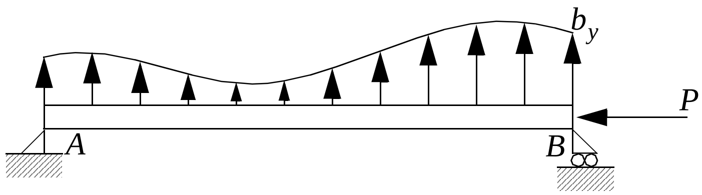
$F = -P$

$\beta = 0$

$V - v,_x P = 0$

Example

$b_x = m = 0$
D_B, D_S, D_T = constant



$$F,_x + b_x = 0 \;\rightarrow\; F(x) = \text{constant} = -P$$

$$D_S\left(u,_x + \frac{1}{2}v,_x^2\right) = -P$$

$$u,_x = -\frac{P}{D_S} - \frac{1}{2}v,_x^2$$

$$u(x) - u(A) = -\frac{Px}{D_S} - \frac{1}{2}\int_0^x v,_x^2 dx$$

$$M = D_B\beta,_x$$
$$M,_x = D_B\beta,_{xx}$$
$$M,_x + V + m = 0$$
$$D_B\beta,_{xx} + V = 0$$
$$V = -D_B\beta,_{xx}$$
$$\frac{V}{D_T} = v,_x - \beta \;\rightarrow\; v,_x = \beta + \frac{V}{D_T} = \beta - \frac{D_B}{D_T}\beta,_{xx}$$

$$V,_x + (Fv,_x),_x + b_y = 0$$
$$-D_B\beta,_{xxx} - Pv,_{xx} + b_y = 0$$
$$-D_B\beta,_{xxx} - P\left(\beta,_x - \frac{D_B}{D_T}\beta,_{xxx}\right) + b_y = 0$$
$$D_B\beta,_{xxx}\left(\frac{P}{D_T} - 1\right) - P\beta,_x + b_y = 0$$
$$\beta,_{xxx} + \frac{P}{D_B\left(1 - \frac{P}{D_T}\right)}\beta,_x = \frac{b_y}{D_B\left(1 - \frac{P}{D_T}\right)}$$

Define $$\mu^2 = \frac{P}{D_B\left(1 - \frac{P}{D_T}\right)}$$

Solving for β and then v, starting with

$$\beta,_{xxx} + \mu^2\beta,_x = \frac{\mu^2}{P}b_y$$

$$\beta = C_2 + \mu\left(1 - \frac{P}{D_T}\right)(-C_4\sin\mu x + C_5\cos\mu x) + \beta_{part}$$

$$v = \int\left(\beta - \frac{D_B}{D_T}\beta,_{xx}\right)dx$$

$$v = C_2 x + C_3 + C_4\cos\mu x + C_5\sin\mu x + v_{part}$$

$$M = D_B\beta,_x = P(-C_4\cos\mu x - C_5\sin\mu x) + D_B(\beta_{part}),_x$$

$$V = -D_B\beta,_{xx}$$

$$V = \mu P(-C_4\sin\mu x + C_5\cos\mu x) - D_B(\beta_{part}),_{xx}$$

For $b_y = 0$

@ $x = 0$

$$\left.\begin{aligned} v = 0 &\rightarrow C_3 + C_4 = 0 \\ M = 0 &\rightarrow C_4 = 0 \end{aligned}\right\} \begin{aligned} C_4 &= 0 \\ C_3 &= 0 \end{aligned}$$

@ $x = L$

$$\left.\begin{aligned} v = 0 &\rightarrow C_2 L + C_5\sin\mu L = 0 \\ M = 0 &\rightarrow C_5\sin\mu L = 0 \end{aligned}\right\} C_2 = 0$$

For a non-trivial solution $\sin\mu L = 0$ (ie $\mu L = n\pi$, $n = 1, 2, \ldots$)

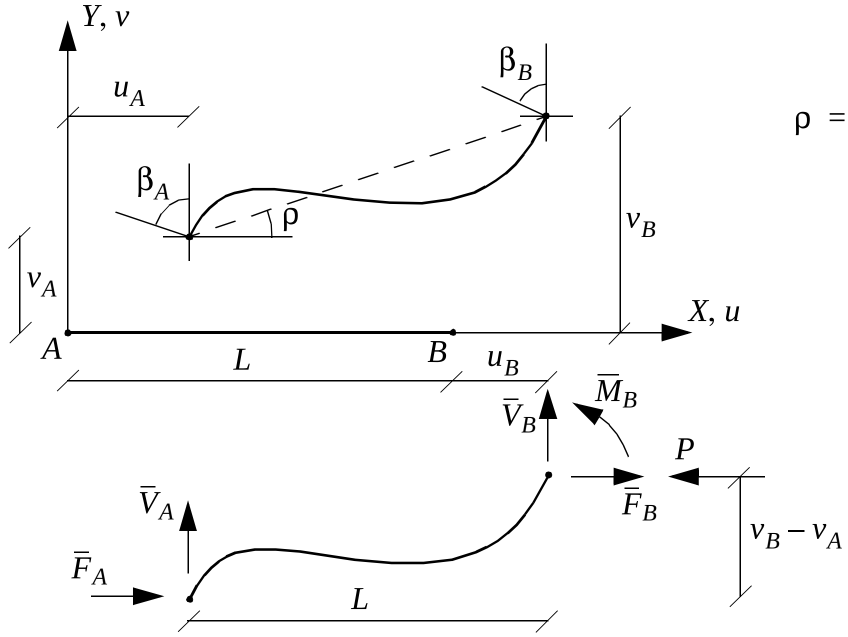
For $n = 1$ $\quad \mu L = \pi \rightarrow \mu = \frac{\pi}{L}$

So

$$\frac{\pi^2}{L^2} = \frac{P}{D_B\left(1 - \frac{P}{D_T}\right)}$$

$$P_{cr} = \frac{\frac{\pi^2 D_B}{L^2}}{1 + \frac{\pi^2 D_B}{L^2 D_T}}$$

5.4 Member Relations Geometrically Non-Linear Case



$$\rho = \frac{v_B - v_A}{L}$$

$$(\mu L)^2 = \frac{PL^2}{D_B} = \lambda^2$$

$$D_B = EI$$

$$D = 2(1 - \cos\mu L) - \mu L \sin\mu L$$

$$D\phi_1 = \mu L(\sin\mu L - \mu L\cos\mu L)$$

$$D\phi_2 = \mu L(\mu L - \sin\mu L)$$

$$\phi_3 = \phi_1 + \phi_2$$

Assumption: Neglect transverse shear deformation wrt bending deformation

$$\underline{u}_A = \begin{bmatrix} u_A \\ v_A \\ \beta_A \end{bmatrix} \qquad \underline{u}_B = \begin{bmatrix} u_B \\ v_B \\ \beta_B \end{bmatrix}$$

$$\underline{\bar{F}}_A = \begin{bmatrix} \bar{F}_A \\ \bar{V}_A \\ \bar{M}_A \end{bmatrix} \qquad \underline{\bar{F}}_B = \begin{bmatrix} \bar{F}_B \\ \bar{V}_B \\ \bar{M}_B \end{bmatrix}$$

5.4.1 Member Equations

$$u_B = u_A - \frac{PL}{D_S} - \int\frac{1}{2}v,_x^2 dx$$

Set $$e_r = \frac{1}{L}\int\frac{1}{2}v,_x^2 dx$$

$$u_B = u_A - \frac{PL}{D_S} - e_r L$$

$$\bar{F}_B = \frac{D_S}{L}(u_B - u_A + e_r L) = -P \text{ (member in compression)}$$

$$\bar{M}_B = \frac{D_B}{L}(\phi_1\beta_B + \phi_2\beta_A - \phi_3\rho)$$

$$\bar{M}_A = \frac{D_B}{L}(\phi_1\beta_A + \phi_2\beta_B - \phi_3\rho)$$

$$\bar{V}_B = -\frac{D_B\phi_3}{L^2}(\beta_B + \beta_A - 2\rho) - P\rho$$

$$\bar{V}_A = -\bar{V}_B$$

Write member equations as

$$\begin{bmatrix}\bar{F}_B\\ \bar{V}_B\\ \bar{M}_B\end{bmatrix} = \begin{bmatrix} D_S/L & 0 & 0\\ 0 & \frac{2D_B\phi_3}{L^3} & -\frac{D_B\phi_3}{L^2}\\ 0 & -\frac{D_B\phi_3}{L^2} & \frac{D_B\phi_1}{L}\end{bmatrix}\begin{bmatrix}u_B\\ v_B\\ \beta_B\end{bmatrix} + \begin{bmatrix} -D_S/L & 0 & 0\\ 0 & -\frac{2D_B\phi_3}{L^3} & -\frac{D_B\phi_3}{L^2}\\ 0 & \frac{D_B\phi_3}{L^2} & \frac{D_B\phi_2}{L}\end{bmatrix}\begin{bmatrix}u_A\\ v_A\\ \beta_A\end{bmatrix} + \begin{bmatrix}D_S e_r\\ -P\rho\\ 0\end{bmatrix}$$

$$\underline{\bar{F}}_B \qquad \underline{k}_{BB} \qquad \underline{u}_B \qquad \underline{k}_{BA} \qquad \underline{u}_A \qquad \underline{F}_B^{(r)}$$

If $\underline{\bar{F}}_B^{(i)}$ = Forces due to span loads

$$\underline{\bar{F}}_B = \underline{k}_{BB}\underline{u}_B + \underline{k}_{BA}\underline{u}_A + \underline{\bar{F}}_B^{(r)} + \underline{\bar{F}}_B^{(i)}$$

Likewise

$$\begin{bmatrix}\bar{F}_A\\ \bar{V}_A\\ \bar{M}_A\end{bmatrix} = \begin{bmatrix} -D_S/L & 0 & 0\\ 0 & -\frac{2D_B\phi_3}{L^3} & \frac{D_B\phi_3}{L^2}\\ 0 & -\frac{D_B\phi_3}{L^2} & \frac{D_B\phi_2}{L}\end{bmatrix}\begin{bmatrix}u_B\\ v_B\\ \beta_B\end{bmatrix} + \begin{bmatrix} D_S/L & 0 & 0\\ 0 & \frac{2D_B\phi_3}{L^3} & \frac{D_B\phi_3}{L^2}\\ 0 & \frac{D_B\phi_3}{L^2} & \frac{D_B\phi_1}{L}\end{bmatrix}\begin{bmatrix}u_A\\ v_A\\ \beta_A\end{bmatrix} + \begin{bmatrix}-D_S e_r\\ P\rho\\ 0\end{bmatrix}$$

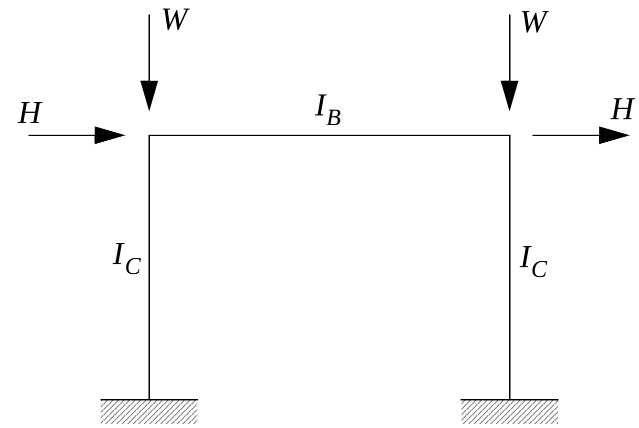
$$\underline{\bar{F}}_B \qquad \underline{k}^T{}_{BA} \qquad \underline{u}_B \qquad \underline{k}_{AA} \qquad \underline{u}_A \qquad \underline{F}_A^{(r)}$$

$$\underline{\bar{F}}_A = \underline{k}_{BA}^T\underline{u}_B + \underline{k}_{AA}\underline{u}_A + \underline{\bar{F}}_A^{(r)} + \underline{\bar{F}}_A^{(i)}$$

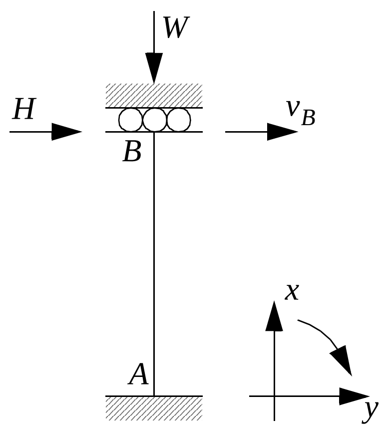
Note: This formulation is wrt local frame and must be transformed to use global frame

5.5 Applications

1. A Frame



Consider $I_B = \infty$



Boundary Conditions

$\beta_A = \beta_B = 0 \quad v_A = 0$

$\rho = \dfrac{v_B}{L} \qquad v_B = v$

$P = W \qquad \bar{V}_B = H$

End actions at B

$$\bar{V}_B = \frac{2D_B\phi_3}{L^3}v - W\rho = \left\{\frac{2D_B\phi_3}{L^3} - \frac{W}{L}\right\}v = H$$

$$\bar{M}_B = -\frac{D_B\phi_3}{L^2}v$$

Write $\bar{V}_B = H = kv$

$$k = \frac{12D_B}{L^3}\left\{\frac{\phi_3}{6} - \frac{WL^2}{12D_B}\right\}$$

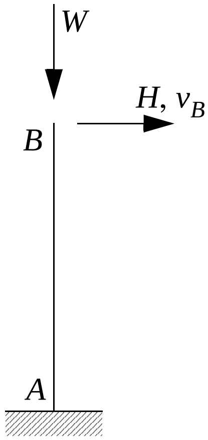
For $W = 0$

$$\phi_3 = 6 \qquad k = \frac{12D_B}{L^3}$$

If we apply $W = P_{cr} = \pi^2 D_B / L^2 \qquad \mu L = \pi \qquad \phi_3 = \pi^2/2$

$$k = \frac{12D_B}{L^3}\left\{\frac{\phi_3}{6} - \frac{W}{P_{cr}}\frac{\pi^2}{12}\right\} = 0 \text{ for } W = P_{cr}$$

2 Cantilever



Boundary Conditions

$$\beta_A = v_A = 0$$

$$M_B = 0$$

$$V_B = H$$

$$P = W$$

Solution

$$\bar{M}_B = \phi_1 \beta_B - \phi_3 \frac{v_B}{L} = 0$$

$$\bar{V}_B = H = -\frac{D_B \phi_3}{L^2}\left\{\beta_B - 2\frac{v_B}{L}\right\} - P\frac{v_B}{L} = k_H v_B$$

$$k_H = -\frac{D_B \phi_3}{L^3}\left\{2 - \frac{\phi_3}{\phi_1}\right\} - \frac{P}{L}$$

Note, from elementary mechanics for a cantilevered column

$$P_{cr} = \frac{\pi^2 D_B}{(2L)^2} = \frac{\pi^2 D_B}{4L^2}$$

Then

If $W = 0$

$$P = 0 \qquad \phi_1 = 4 \qquad \phi_2 = 2 \qquad \phi_3 = 6$$

$$k_H = \frac{3D_B}{L^3} \leftarrow \text{ Linear case}$$

If $W \neq 0$

$$k_H = \frac{D_B}{L^3}\left\{2\phi_3 - \frac{\phi_3^2}{\phi_1} - \frac{P}{(D_B/L^2)}\right\}$$

At the critical load

$$\mu L = \frac{\pi}{2} \qquad D = 2 - \frac{\pi}{2} \qquad D\phi_1 = \frac{\pi}{2} \qquad D\phi_3 = \left(\frac{\pi}{2}\right)^2$$

$$k_H = 0$$

At $P = \frac{1}{2}P_{cr}$

$$\mu L = \frac{\pi}{2}\left(\frac{1}{\sqrt{2}}\right) \cong 1.11$$

$$\cos\mu L \cong 0.445$$

$$\sin\mu L \cong 0.9$$

$$\phi_1 \cong 4$$

$$\phi_3 \cong 6$$

$$k_H = \frac{D_B}{L^3}\left\{2\phi_3 - \frac{\phi_3^2}{\phi_1} - \frac{4P}{\pi^2}\right\} \cong 1.77\frac{D_B}{L^3}$$

So, about a 40% reduction in stiffness due to axial loading.

3 Simply Supported Beam-Column

Boundary Conditions

$$M_A = 0 \qquad v_A = 0$$

$$M_B = M_B{*} \qquad v_B = 0$$

Solution

$$M_A = 0 = \phi_1\beta_A + \phi_2\beta_B$$

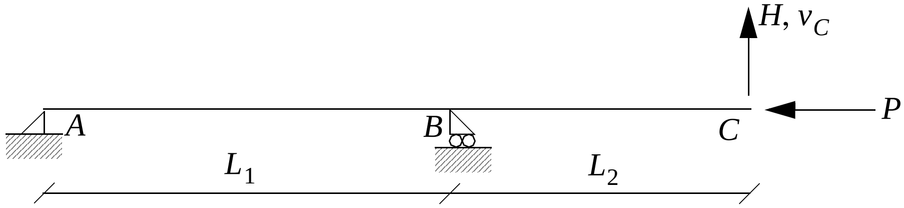
$$\beta_A = -\frac{\phi_2}{\phi_1}\beta_B$$

$$M_B = \frac{D_B}{L}\left(\phi_1 - \frac{\phi_2^2}{\phi_1}\right)\beta_B = k_B\beta_B$$

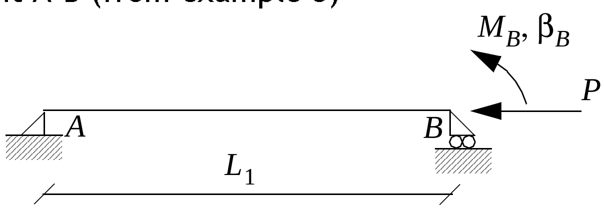
If $P = 0$

$$k_B = 3\frac{D_B}{L} \text{ (Linear case)}$$

4 Multi Span Beam-Column

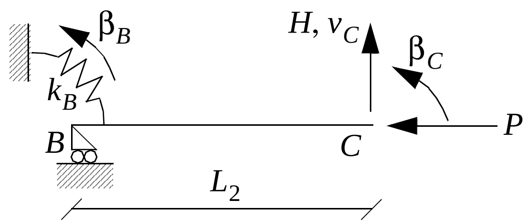


Segment A-B (from example 3)



$$M_B = k_B \beta_B$$

Segment B-C



$$\rho = \frac{v_C}{L_2}$$

Boundary Conditions

$$M_C = 0$$

$$v_B = 0$$

$M_B = -k_B \beta_B$ (be careful of sign convention)

$$V_C = H$$

Solution

i) $$M_B = \frac{D_B}{L_2}\left(\phi_1 \beta_B + \phi_2 \beta_C - \phi_3 \frac{v_C}{L_2}\right) = -k_B \beta_B$$

ii) $$M_C = \frac{D_B}{L_2}\left(\phi_1 \beta_C + \phi_2 \beta_B - \phi_3 \frac{v_C}{L_2}\right) = 0$$

Use i) and ii) to solve β_B and β_C in terms of v_C

$$\beta_B\left\{k_B + \frac{D_B}{L_2}\phi_1\right\} + \frac{D_B}{L_2}\phi_2 \beta_C = \frac{D_B}{L_2}\phi_3 \frac{v_C}{L_2}$$

$$\frac{\phi_2}{\phi_1}\beta_B + \beta_C = \frac{\phi_3}{\phi_1}\frac{v_C}{L_2}$$

Then substitute in expression for H

$$H = \frac{D_B \phi_3}{L_2^2}\left\{-\beta_C - \beta_B + 2\frac{v_C}{L_2}\right\} - P\frac{v_C}{L_2}$$

5.5 Incremental Formulation

5.5.1 Some simplifications

1.

$$e_r = \frac{1}{2L}\int v_{,x}^2 dx$$

Take $v_{,x} \cong \frac{v_B - v_A}{L} = \rho$

Then

$$e_r \cong \frac{1}{2L}(\rho^2 L) = \frac{\rho^2}{2}$$

2. Assume ϕ 's are constant during incremental motion. This results in $\underline{k}_{BB}$, $\underline{k}_{AA}$, and $\underline{k}_{BA}$ constant. (ie $d\underline{k} = 0$)

3. Consider small increment, ie work with first order change.

$$d\bar{\underline{F}}_B \cong \underline{k}_{BB}\Delta\underline{u}_B + \underline{k}_{BA}\Delta\underline{u}_A + d\bar{\underline{F}}_B^{(r)} + d\bar{\underline{F}}_B^{(i)}$$

$$d\bar{\underline{F}}_A = \underline{k}_{BA}^T\Delta\underline{u}_B + \underline{k}_{AA}\Delta\underline{u}_A + d\bar{\underline{F}}_A^{(r)} + d\bar{\underline{F}}_A^{(i)}$$

$$d\bar{\underline{F}}_A^{(r)} = -d\bar{\underline{F}}_B^{(r)}$$

$$d\bar{\underline{F}}_B^{(r)} = d\bar{\underline{F}}^{(r)}$$

$$\underline{F}^{(r)} = \begin{bmatrix} D_S e_r \\ F_B\rho \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{D_S}{2}\rho^2 \\ F_B\rho \\ 0 \end{bmatrix}$$

$$F_B = \frac{D_S}{L}(u_B - u_A) + D_S e_r$$

operating on $d\bar{\underline{F}}^{(r)}$

$$d\bar{\underline{F}}^{(r)} = \begin{bmatrix} D_S\rho d\rho \\ F_B d\rho + dF_B\rho \\ 0 \end{bmatrix}$$

$$d\rho = \frac{\Delta v_B - \Delta v_A}{L}$$

$$dF_B = \frac{D_S}{L}(\Delta u_B - \Delta u_A) + D_S\rho\left(\frac{\Delta v_B - \Delta v_A}{L}\right)$$

Then

$$d\underline{\bar{F}}^{(r)} = \begin{bmatrix} \dfrac{D_S\rho}{L}(\Delta v_B - \Delta v_A) \\ \dfrac{F_B}{L}(\Delta v_B - \Delta v_A) + \dfrac{D_S\rho}{L}(\Delta u_B - \Delta u_A) + \dfrac{D_S\rho^2}{L}(\Delta v_B - \Delta v_A) \\ 0 \end{bmatrix}$$

$$d\underline{\bar{F}}^{(r)} = \underline{K}_R(\Delta u_B - \Delta u_A)$$

$$\underline{K}_R = \begin{bmatrix} 0 & \dfrac{D_S\rho}{L} & 0 \\ \dfrac{D_S\rho}{L} & \dfrac{F_B}{L} + \dfrac{D_S\rho^2}{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5.5.2 Summarizing

$$d\underline{F}_B \cong (\underline{k}_{BB} + \underline{K}_R)\Delta\underline{u}_B + (\underline{k}_{BA} - \underline{K}_R)\Delta\underline{u}_A + d\underline{\bar{F}}_B^{(i)}$$

$$d\underline{\bar{F}}_A \cong (\underline{k}_{BA} + \underline{K}_R)^T\Delta\underline{u}_B + (\underline{k}_{AA} + \underline{K}_R)\Delta\underline{u}_A + d\underline{\bar{F}}_A^{(i)}$$

These two expressions define the tangent stiffness for a member. One usually retains the non-linear terms only when the axial force is compressive since the stiffness is reduced. A tensile axial force increases the stiffness.

5.5.3 Linearized Stability Analysis

1. Take $\rho = 0$ in $\underline{K}_R$.

2. Find axial forces with linear analysis (ie ignore $\underline{K}_R$.) Evaluate ϕ's.

3. Test determinant of tangent stiffness matrix.