LECTURE # 34

1.060 ENGINEERING MECHANICS II

THE KINEMATIC WAVE

In Lecture # 33 we found that unsteady flow in a prismatic channel could be described by a kinematic wave equation

$$\frac{\partial h}{\partial t} + C_k \frac{\partial h}{\partial x} = 0$$

Wide Rectangular Chan.

where

$$C_k = \frac{\partial Q/\partial h}{b_s} = \begin{cases} \frac{5}{3} V = \frac{5}{3}\left(\frac{1}{n}\sqrt{S_0}\right) h^{2/3} \ (M) \\ \frac{3}{2} V = \frac{3}{2}\left(C\sqrt{S_0}\right) h^{1/2} \ (D\text{-}W) \end{cases}$$

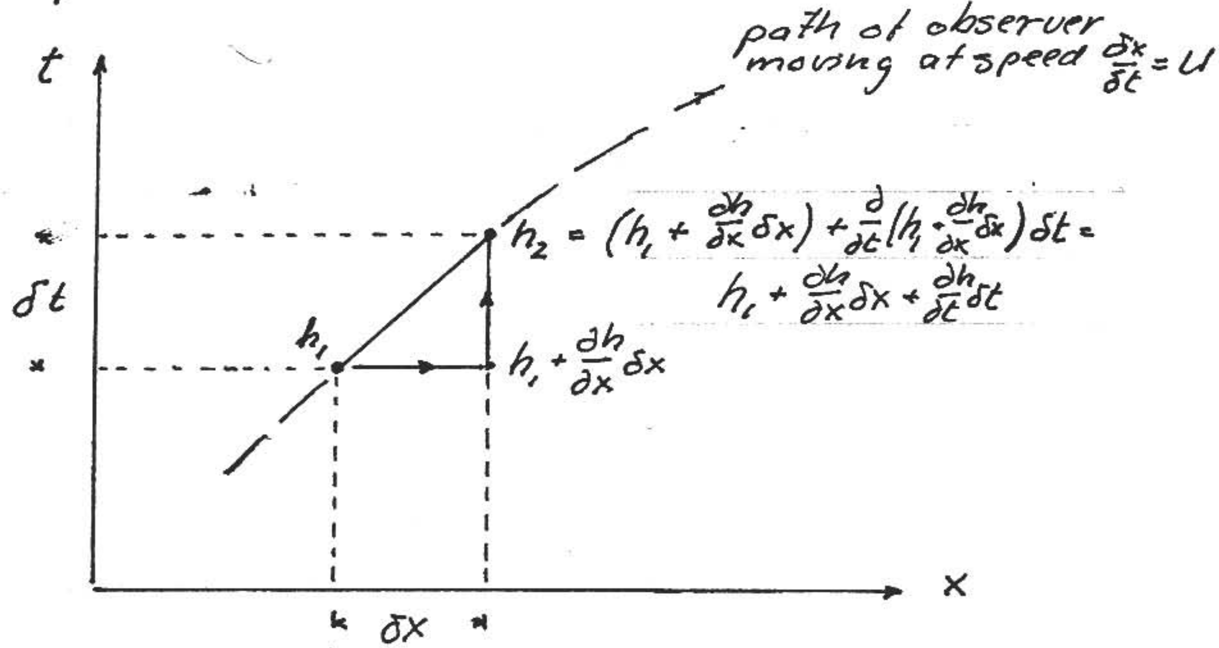
If h were approximated by a constant value, assuming that $h = h_0 + \eta$ with $\eta \ll h_0$, $C_k \cong C_{k0}$ = constant and the kinematic wave equation becomes

$$\frac{\partial h}{\partial t} + C_{k0} \frac{\partial h}{\partial x} = 0$$

whose solution is a wave of constant shape traveling down the channel at a constant speed C_{k0}:

$$h(x,t) \cong h(x - C_{k0} t)$$

We may visualize this solution by considering the rate of change in depth experienced by an observer moving along the x-axis at a velocity U. The position of this observer as a function of time may be represented by her path in the xt-plane



From the sketch above it follows that

$$h_2 - h_1 = \frac{\partial h}{\partial t}\delta t + \frac{\partial h}{\partial x}\delta x$$

or

$$\frac{h_2 - h_1}{\delta t} = \frac{Dh}{Dt} = \text{rate of change in 'h' seen by observer moving at speed } \delta x/\delta t = U = \frac{\partial h}{\partial t} + \left(\overset{U}{\frac{\delta x}{\delta t}}\right)\frac{\partial h}{\partial x}$$

Comparing this expression to our kinematic wave equation we have

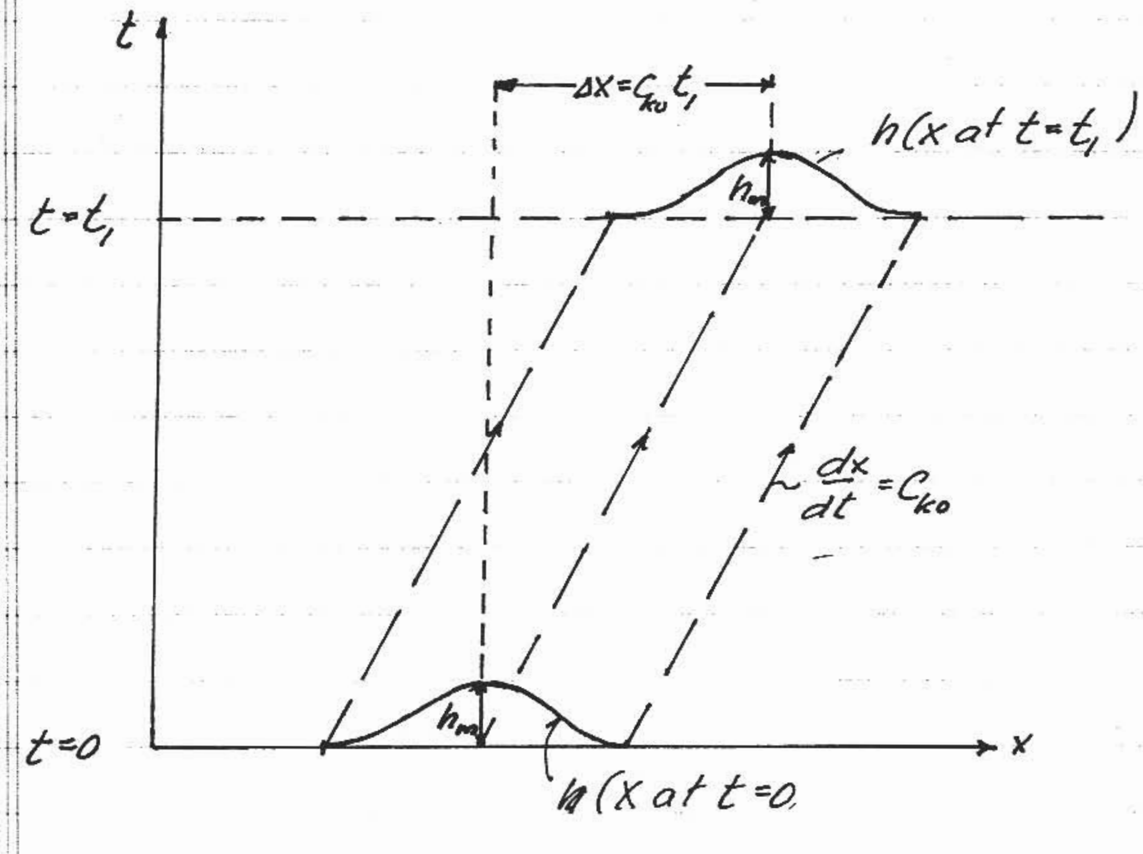
$$\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + C_k \frac{\partial h}{\partial x} = 0 = \text{rate of change in 'h' seen by an observer moving at speed } dx/dt = C_k \text{ along x-axis.}$$

Since

$$\frac{Dh}{Dt} = 0 \quad \Rightarrow \quad h = \text{constant}$$

as far as an observer moving at a velocity $dx/dt = c_k$ along the x-axis.

If $c_k = c_{ko}$ = constant this means that any point along the initial profile of h, e.g. $h = h_{max}$, moves down the channel at a speed equal to c_{ko}. This is illustrated in the xt-plane below and is exactly the behavior of a wave of constant form moving in the x-direction at speed c_{ko} (as implied by $h(x,t) = h(x - c_{ko}t)$).



This is, however, not all we can get from this interpretation of

$$\frac{\partial h}{\partial t} + C_k \frac{\partial h}{\partial x} = \frac{Dh}{Dt} = 0 \quad \text{along paths defined by } \frac{dx}{dt} = C_k$$

The equation is the same, i.e.

$$\frac{Dh}{Dt} = 0 \Rightarrow h = \text{constant}$$

along paths (known as "characteristics") defined by

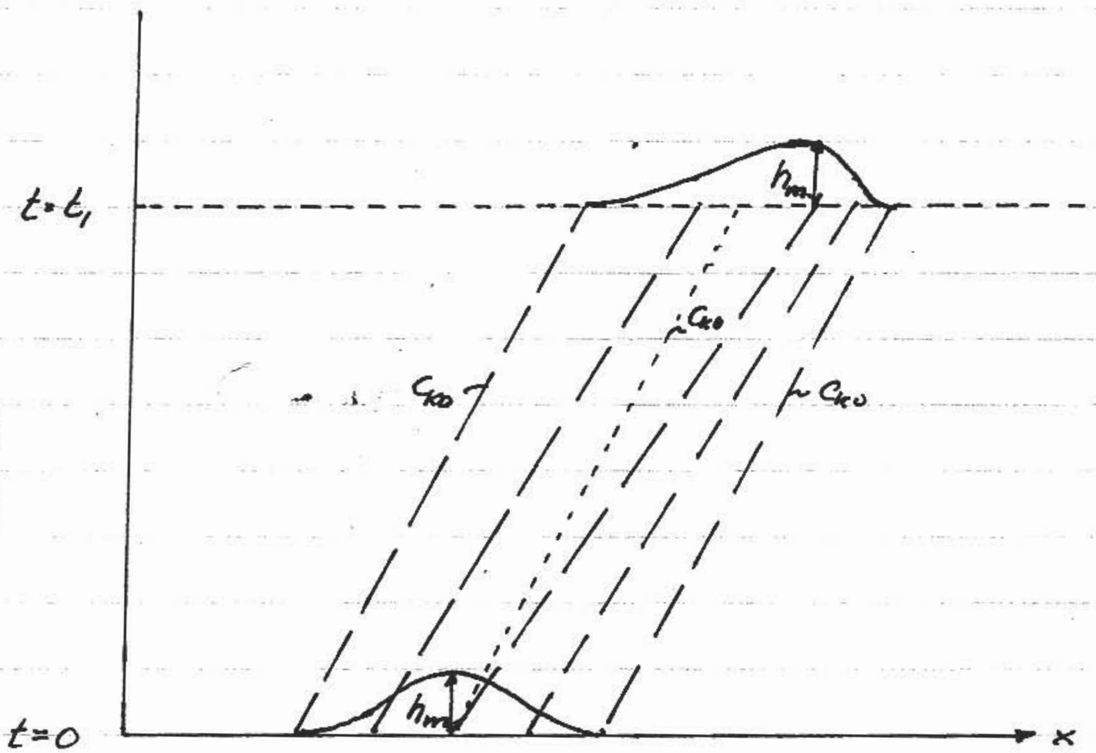
$$\frac{dx}{dt} = C_k = \frac{5}{3}\left(\frac{1}{n}\sqrt{S_0}\right) h^{2/3} \quad (M)$$

But if h is constant along $dx/dt = C_k$ and since C_k depends only on h, then C_k is constant and the paths defined by $dx/dt = C_k$ are straight lines in the xt-plane. These lines are parallel if h is replaced by h_0 = constant and $C_k = C_{k0}$. However, if we allow h to vary the velocity of each observer to follow the location of a given depth is constant but depends on the depth she follows, e.g. the depth at $t=0$, i.e.

$$\frac{dx}{dt} = C_k = \frac{5}{3}\left(\frac{1}{n}\sqrt{S_0}\right)\left[h\,(x \text{ at } t=0)\right]^{2/3}$$

and we note that $\frac{dx}{dt}$ is larger when h is larger

In this approximation the evolution of an initial symmetric wave is readily visualized in the xt-plane below



As the wave moves down the channel the crest, where $h = h_m$ is the largest, moves the fastest. Thus, the wave becomes forward-leaning (time of rise decreases) with a steeper front and flatter back (time of ~~fall~~ increases). If allowed to go on the paths of two observers will evantually coincide. When this happens the water surface will become double-valued, i.e. the front of the wave will have a jump in elevation. Since our approximation assumed that $|dh/dx| < S_0$ this prediction violates our assumptions and can therefore not be considered realistic.

We have seen in Lecture #33 that the neglect of the free surface slope term in the momentum equation placed the most severe restriction on the validity of our simple kinematic wave model. Since we found that this simple model suggested that the slope of the free surface might become large as time goes on, it would be nice if we were able to include this term in our model.

To do this we take

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_f} = K(h)\sqrt{S_0 - \partial h/\partial x}$$

and introduce this expression in the continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial h}\frac{\partial h}{\partial t} = \frac{\partial Q}{\partial x} + b_s\frac{\partial h}{\partial t} = 0$$

by obtaining

$$\frac{\partial Q}{\partial x} = \sqrt{S_0 - \partial h/\partial x}\;\frac{\partial K}{\partial x} + K\frac{1}{2}\frac{-\partial^2 h/\partial x^2}{\sqrt{S_0 - \partial h/\partial x}} =$$

$$\left(\sqrt{S_0 - \partial h/\partial x}\;\frac{\partial K}{\partial h}\right)\frac{\partial h}{\partial x} - \frac{K}{2\sqrt{S_0 - \partial h/\partial x}}\frac{\partial^2 h}{\partial x^2}$$

The resulting governing equation now reads:

$$\frac{\partial h}{\partial t} + C_{kD}\frac{\partial h}{\partial x} = D_k \frac{\partial^2 h}{\partial x^2}$$

where

$$C_{kD} = \frac{\partial k/\partial h}{b_s}\sqrt{S_o - \frac{\partial h}{\partial x}}$$

$$D_k = \frac{K}{2b_s\sqrt{S_o - \partial h/\partial x}}.$$

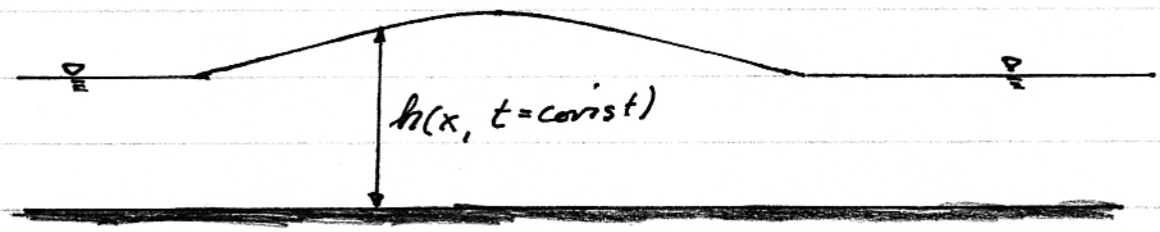
This approximation for the kinematic wave is known as the "Diffusion Analogy" since the term on the right hand side has the appearence of a "diffusion" and results in a spreading (diffusion) of the wave form and a decay of the maximum value of h as time goes on.

We may again write this equation in its characteristic form, i.e.

$$\frac{Dh}{Dt} = D_k \frac{\partial^2 h}{\partial x^2} \quad \text{along } \frac{dx}{dt} = C_{kD} \text{ (characteristics)}$$

In this case the determination of the characteristics is not that simple since C_{kD} no longer is a function of h alone but also of $\partial h/\partial x$. Even if it were only a function of h, we would still not be able to pre-determine the characteristics, once $Dh/Dt \neq 0$ and the depth therefore changes along a characteristic making this a curved path rather than a straight line.

Without going into any details, we may, however, observe an important feature predicted by the Diffusion Analogy, from the characteristic form of the equation.



Near the crest of a flood-wave we have

$$\frac{\partial^2 h}{\partial x^2} < 0 \qquad \text{near crest}$$

and therefore

$$\frac{Dh}{Dt} = D_k \frac{\partial^2 h}{\partial x^2} < 0 \qquad \text{near crest}$$

Consequently, the crest elevation decreases as the flood-wave travels down the channel. Thus, the diffusion analogy leads to a "subsidence" of the flood-wave as it moves down the river.