1.022 Introduction to Network Models

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Lecture 21

Outline

- Decisions, utility maximization
- Games and Strategies
- Best Responses and Dominant Strategies (Split or Steal)
- Dominance Solvability (Split, Steal, or Quit)
- Nash Equilibrium in pure strategies (coordination game)
- Nonexistence of pure strategy Nash equilibria
- Multiplicity of Nash equilibria

Reading:

- Easley-Kleinberg, Ch. 6.1-6.6, 6.10 (A&B).

Golden Balls: Split or Steal?

- Two TV show contestants simultaneously pick to either split or steal the prize.

Player 1 / Player 2	Split	Steal
Split	$(7, 7)$	$(0, 14)$
Steal	$(14, 0)$	$(0, 0)$

- Here the first number is the payoff to player 1 and the second number is the payoff to player 2. More formally, the cell indexed by row x and column y contains a pair, (a, b) where $a = u_1(x, y)$ and $b = u_2(x, y)$.
- What will the outcome of this game be? How would you play?
- Regardless of what the other player does, playing **Steal** is better for each player. Does it help to talk your opponent out of playing **Steal**?
- This is also known as a "prisoner's dilemma".

Motivation

- In the context of networked systems (social, communication, trasportation, ...) agents make a variety of choices.
- For example:
  - What kind of information to share with others you are connected to.
  - How to evaluate information obtained from friends, neighbors, coworkers and media.
  - Whether to trust and form friendships.
  - Which of the sellers in your neighborhood to use.
  - Which websites to visit.
  - How to map your drive in the morning (or equivalently how to route your network traffic).
- In all of these cases, *interactions* with other agents you are connected to affect your payoff, well-being, utility.
- How to make decisions in such situations?
- $\rightarrow$ "multiagent decision theory" or game theory.

Rationality and Decision-Making

- Powerful working hypothesis in economics: *individuals act rationally* in the sense of choosing the option that gives them higher "payoff".
  - Payoff here need not be monetary payoff. Social and psychological factors influence payoffs and decisions.
  - People do not literally maximize utility, but they often act as if they do.
  - Can be hard to write down the "right" utility function.
  - Nevertheless, the rational decision-making paradigm is useful because it provides us with a (testable) theory of economic and social decisions.
- **Classical view** of game theory: the game describes everything we need to know and the game can be studied in isolation.
- **Instrumental view** of game theory: the game as an imperfect model of the real world, but a more realistic model isn't necessarily a better model.

Strategic Form Games

- We study the games in which all of the participants act simultaneously and without knowledge of other players' actions. Such games are referred to as strategic form games—or as *normal form* games or *matrix* games.
- For each game, we have to define
  1. The set of players.
  2. The strategies.
  3. The payoffs.
- More generally, we also have to define the *game form*, which captures the order of play (e.g., in chess) and information sets (e.g., in asymmetric information or incomplete information situations). But in strategic form games, play is simultaneous, so no need for this additional information.

More formally:

Strategic Form Game: A strategic forms game is a triplet $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ such that:

- $\mathcal{I}$ is a finite set of players, i.e., $\mathcal{I} = \{1, \dots, I\}$;
- S_i is the set of available actions for player i;
- $s_i \in S_i$ is an action for player i;
- $u_i : S \to \mathbb{R}$ is the payoff (utility) function of player i where $S = \prod_i S_i$ is the set of all action profiles.

In addition, we use the notation:

- $s_{-i} = [s_j]_{j \neq i}$: vector of actions for all players except i.
- $S_{-i} = \prod_{j \neq i} S_j$ is the set of all action profiles for all players except i
- $(s_i, s_{-i}) \in S$ is a **strategy profile**, or outcome.

Strategies

- In game theory, a strategy is a complete description of how to play a game.
- It requires full contingent planning. If instead of playing the game yourself, you had to delegate the play to a "computer" with no initiative, then you would have to spell out a full description of how the game would be played in every contingency.
- For example, in chess, this would be a hard task (though in some simpler games, it can be done more easily).
- Thinking in terms of strategies is important.
- But in strategic form games, there is no difference between an action and a pure strategy, and we will use them interchangeably. (not valid for mixed strategies)

Finite Strategy Spaces

- When the strategy space is finite, and the number of players and actions is small, a game can be represented in matrix form.
- Recall that the cell indexed by row x and column y contains a pair, (a, b) where $a = u_1(x, y)$ and $b = u_2(x, y)$.

Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- This game represents pure conflict in the sense that one player's utility is the negative of the utility of the other player. Thus, it is a **zero sum game.**

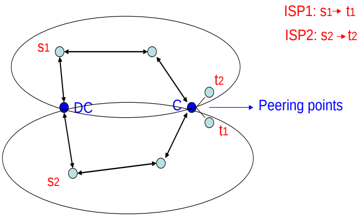
Dominant Strategies

- Example: Prisoner's Dilemma.
  - Two people arrested for a crime, placed in separate rooms, and the authorities are trying to extract a confession against each other.

prisoner 1 / prisoner 2	Betray	Stay silent
Betray	$(-4,-4)$	$(-1,-5)$
Stay silent	$(-5,-1)$	$(-2,-2)$

- What will the outcome of this game be?
- Regardless of what the other player does, playing "Betray" is better for each player.
- The action "Betray" **strictly dominates** the action "Stay silent"
- Prisoner's dilemma paradigmatic example of a self-interested, rational behavior not leading to jointly *(socially) optimal result*.

- Consider two Internet service providers that need to send traffic to each other
- Assume that the unit cost along a link (edge) is 1



- This situation can be modeled by the "Prisoner's Dilemma" payoff matrix.

ISP 1 / ISP 2	Hot potato routing	Minimum distance routing
Hot potato routing	$(-4,-4)$	$(-1,-5)$
Minimum distance routing	$(-5,-1)$	$(-2,-2)$

Dominant Strategy Equilibrium

- Compelling notion of equilibrium in games would be dominant strategy equilibrium, where each player plays a dominant strategy.

Dominant Strategy: A strategy $s_i \in S_i$ is *dominant* for player i if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \qquad \text{for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}.$$

Dominant Strategy Equilibrium: A strategy profile s^* is the dominant strategy equilibrium if for each player i, s_i^* is a dominant strategy.

- These notions could be defined for *strictly* dominant strategies as well.

- Show that in the prisoner's dilemma game, "betray, betray" is a dominant strategy equilibrium.

Example: Split or Steal?

Player 1 / Player 2	Split	Steal
Split	$(7,7)$	$(0,14)$
Steal	$(14,0)$	$(\mathbf{0},\ \mathbf{0})$

It's easy to check that (**Steal, Steal**) is a dominant strategy equilibrium.

Player 1 / Player 2	Split	Steal	Quit
Split	$(7,7)$	$(0,14)$	$(7,-10)$
Steal	$(14,0)$	$(0,0)$	$(0,-10)$
Quit	$(-10,7)$	$(-10,0)$	$(-10,-10)$

It's easy to check that there is no dominant strategy equilibrium. But **Quit** is not a "rational" strategy. Why?

Strictly Dominated Strategy: A strategy $s_i \in S_i$ is *strictly dominated* for player i if there exists some $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

Example: Split, Steal, or Quit

Player 1 / Player 2	Split	Steal	Quit
Split	$(7,7)$	$(0,14)$	$(7,-10)$
Steal	$(14,0)$	$(0,0)$	$(0,-10)$
Quit	$(-10,7)$	$(-10,0)$	$(-10,-10)$

- **Quit** is a strictly dominated strategy for both players.
- No "rational" player would choose **Quit**. Hence if Player 1 is certain that Player 2 is rational, then he can eliminate the latter's **Quit** strategy, and likewise for Player 2.
- Thus after one round of elimination of strictly dominated strategies, we are back to the Split or Steal, which has a *dominant strategy equilibrium*.
- Thus iterated elimination of strictly dominated strategies leads to a unique outcome, (**Steal, Steal**) —such a game is called **dominance solvable**.

A game that is not dominance solvable

Example: Meeting Tom Schelling in New York

Player 1 \ Player 2	library	station
library	$(1,1)$	$(0,0)$
station	$(0,0)$	$(1,1)$

It's easy to check that this game is not dominance solvable, but (**library, library**) and (**station, station**) seems to be quite reasonable strategy profiles.

Pure strategy Nash equilibrium: A pure strategy Nash Equilibrium of a strategic game $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ is a strategy profile $s^* \in S$ such that for all $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- Put differently, the conjectures of the players are *consistent*: each player i chooses s_i^* expecting all other players to choose s_{-i}^*, and each player's conjecture is verified in a Nash equilibrium.

Player 1 \ Player 2	library	station
library	**(1,1)**	(0, 0)
station	(0, 0)	**(1,1)**

Dominant strategy equilibrium ⇒ dominance solvable ⇒ Unique PSNE

Reasoning about Nash Equilibrium

- This has a "steady state" type flavor. In fact, two ways of justifying Nash equilibrium rely on this flavor:
  1. Introspection: what I do must be consistent with what you will do given your beliefs about me, which should be consistent with my beliefs about you,...
  2. Steady state of a learning or evolutionary process.
- A complementary justification: Nash equilibrium is *self-reinforcing*
  - If player 1 is told about player 2's strategy, in a Nash equilibrium she would have no incentive to change her strategy. Think back to Golden Balls example!
- Role of conjectures: let us revisit matching pennies

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

  - Here, player 1 can play heads expecting player 2 to play tails. Player 2 can play tails expecting player 1 to play tails.
  - But these conjectures are not consistent with each other.

Nonexistence of Pure Strategy Nash Equilibria

Pure strategy Nash equilibrium might not always exist. Example: Soccer

Kicker \ Goalie	left	right
left	$(-1,1)$	$(1,-1)$
right	$(1,-1)$	$(-1,1)$

- There is no pure strategy Nash equilibrium: kickers do not always shoot left and goalies do not always jump right...
- ... instead they randomize!

Example: Matching Pennies

Player 1 \ Player 2	heads	tails
heads	$(-1,1)$	$(1,-1)$
tails	$(1,-1)$	$(-1,1)$

- No pure Nash equilibrium.
- How would you play this game?

Multiple Pure Strategy Nash Equilibria

- While non-existence of Pure Strategy Nash Equilibria feels like a problem, in many games, the opposite is true: there are very many pure strategy Nash equilibria.
- Which equilibrium is played is an interesting theoretical and empirical question. Expectations matter for equilibrium selection!

Example: Meeting Tom Schelling in New York.

Player 1 \ Player 2	library	station
library	**(1,1)**	(0, 0)
station	(0, 0)	**(1,1)**

- Experimentally, huge breakdown in coordination for *tiny* payoff asymmetries.[1]

[1] Crawford, V.P. *et al.* (2006) The power of focal points is limited: even minute payoff asymmetry may yield large coordination failures, *American Economic Review*.

Multiple Pure Strategy Nash Equilibria

Example: Battle of the Sexes Game.

Player 1 \ Player 2	ballet	football
ballet	**(1,4)**	$(0,0)$
football	$(0,0)$	**(4,1)**

- This game has two pure Nash equilibria.

Example: Partnership Game.

Player 1 \ Player 2	work hard	shirk
work hard	**(2,2)**	$(-1,1)$
shirk	$(1,-1)$	**(0,0)**

- Also two pure Nash equilibria.