1.017/1.010 Class 16
Testing Hypotheses about a Single Population

Formulating Hypothesis Testing Problems

Hypotheses about a random variable $\boldsymbol{x}$ are often formulated in terms of its distributional properties. Example, if property is a:

Null hypothesis H0: $a = a_0$

Objective of **hypothesis testing** is to decide whether or not to **reject** this hypothesis. Decision is based on estimator $\hat{\boldsymbol{a}}$ of a:

Reject H0: If observed estimate $\hat{a}$ lies in **rejection region** R_{a0} ($\hat{a} \in R_{a0}$)

Do not reject H0: Otherwise ($\hat{a} \notin R_{a0}$)

Select rejection region to obtain desired error properties:

		Test Result	
		Do not reject H0 $\hat{a} \notin R_{a0}$	Reject H0 $\hat{a} \in R_{a0}$
True situation	H0 true	$P(\text{H0}\|\text{H0}) = 1 - \alpha$	$P(\sim\text{H0}\|\text{H0}) = \alpha$ (**Type I Error**)
	H0 false	$P(\text{H0}\|\sim\text{H0}) = \beta$ (**Type II Error**)	$P(\sim\text{H0}\|\sim\text{H0}) = 1 - \beta$

Type I error probability α is called the test **significance level**.

Deriving Hypothesis Rejection Regions for Large Sample Tests

Hypothesis test is often based on a **standardized statistic** that depends on unknown true property and its estimate. Basic concepts are the same as used to derive confidence intervals (see Class 14).

An example is the z **statistic:**

$$\boldsymbol{z}(\hat{\boldsymbol{a}}, a) = \frac{\hat{\boldsymbol{a}} - a}{SD[\hat{\boldsymbol{a}}]}$$

If the estimate is unbiased $E[\boldsymbol{z}] = 0$ and $Var[\boldsymbol{z}] = 1$.

Define a rejection region R_{z0} in terms of z as:

$$R_{z0}: \boldsymbol{z}(\hat{\boldsymbol{a}}, a_0) \le z_L$$
$$\boldsymbol{z}(\hat{\boldsymbol{a}}, a_0) \ge z_U$$

As rejection region grows Type I error increases and Type II error decreases (test is more likely to reject hypothesis).

As rejection region shrinks Type I error decreases and Type II error increases (test is less likely to reject hypothesis)

Usual practice is to **select rejection region to insure that Type I error probability is equal to a specified value α.**

For a **two-sided test** require that Type I error probability is distributed equally between intervals below z_L (probability = $\alpha/2$) and above z_U (probability = $\alpha/2$).

These probabilities are:

$$P[\boldsymbol{z}(\hat{\boldsymbol{a}}, a) \le z_L \mid \mathrm{H0}] = P[\boldsymbol{z}(\hat{\boldsymbol{a}}, a_0) \le z_L] = F_{\boldsymbol{z}}(z_L) = \frac{\alpha}{2}$$

$$P[\boldsymbol{z}(\hat{\boldsymbol{a}}, a) \ge z_U \mid \mathrm{H0}] = P[\boldsymbol{z}(\hat{\boldsymbol{a}}, a_0) \ge z_U] = 1 - F_{\boldsymbol{z}}(z_U) = \frac{\alpha}{2}$$

$$z_L = F_{\boldsymbol{z}}^{-1}\left(\frac{\alpha}{2}\right) \qquad z_U = F_{\boldsymbol{z}}^{-1}\left(1 - \frac{\alpha}{2}\right)$$

For **large samples** $\boldsymbol{z}(\hat{\boldsymbol{a}}, a_0)$ has a **unit normal** distribution. Use the MATLAB function `norminv` to evaluate $F_{\boldsymbol{z}}^{-1}$.

If the definition of $\boldsymbol{z}$ is applied a two-sided rejection region R_{a0} can also be written directly in terms of the estimate $\hat{a}$:

$$R_{a0}: \hat{a} \le a_L = a_0 + F_{\boldsymbol{z}}^{-1}\left(\frac{\alpha}{2}\right) SD[\hat{\boldsymbol{a}}]$$
$$\hat{a} \ge a_U = a_0 + F_{\boldsymbol{z}}^{-1}\left(1 - \frac{\alpha}{2}\right) SD[\hat{\boldsymbol{a}}]$$

p Values

p value is **largest** significance level resulting in acceptance of $H0$.
For a symmetric two-sided rejection region and a large sample:

$$p/2 = 1 - F_z\left[\frac{\hat{a}-a_0}{SD(\hat{\boldsymbol{a}})}\right] \qquad \hat{a} \geq a$$

$$p/2 = F_z\left[\frac{\hat{a}-a_0}{SD(\hat{\boldsymbol{a}})}\right] \qquad \hat{a} \leq a_{00}$$

For large samples use the MATLAB function `normcdf` to compute p from $\hat{a}$ and $SD[\hat{a}]$.

Special Case -- Sample mean

Consider hypothesis about value of population mean $a = E[\boldsymbol{x}]$:

H0: $a = E[\boldsymbol{x}] = a_0$

Base test on sample mean estimator $\boldsymbol{m_x}$. Obtain $SD[\boldsymbol{m_x}]$ from sample standard deviation:

$$SD[\boldsymbol{m_x}] = \frac{SD[\boldsymbol{x}]}{\sqrt{\boldsymbol{N}}} \approx \frac{s_x}{\sqrt{N}}$$

<u>Example: Testing whether mean is significantly different from zero</u>

Suppose $a_0 = 0$, $s_x = 3$, $N = 9$, $m_x = 1.2$ and $\alpha = .05$:

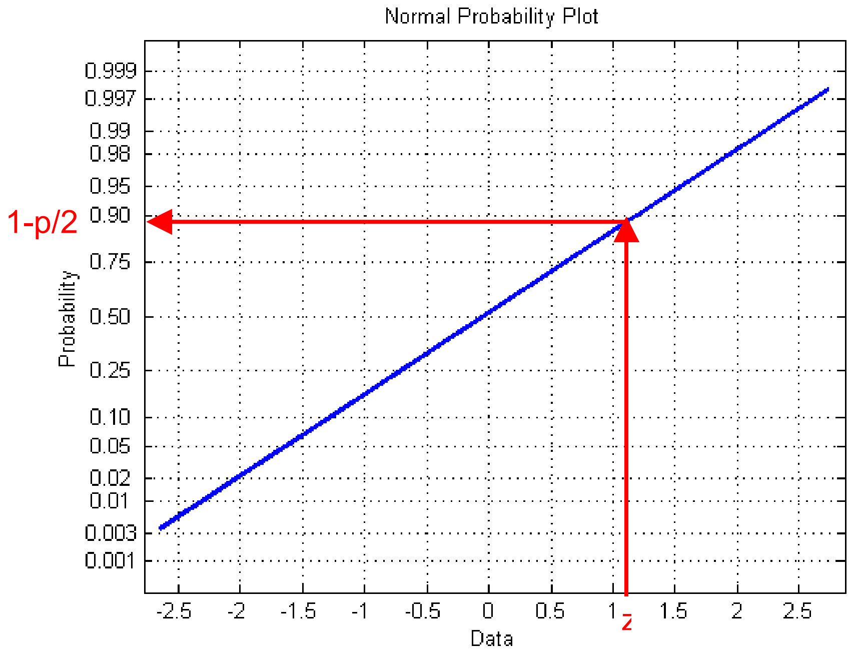
$$R_{a0} : m_x \leq a_L = 0 + F_{\boldsymbol{z}}^{-1}\left(\frac{0.05}{2}\right)\frac{3}{\sqrt{9}} = -1.96$$

$$m_x \geq a_U = 0 + F_{\boldsymbol{z}}^{-1}\left(1 - \frac{0.05}{2}\right)\frac{3}{\sqrt{9}} = +1.96$$

In this case hypothesis is **not rejected** since $m_x = 1.2$ does not lie in R_{a0}. The two-sided p-value is (see plot):

$$1 - p/2 = F_z\left[\frac{m_x - a_0}{s_x / \sqrt{N}}\right] = F_z\left[\frac{1.2-0}{3/\sqrt{9}}\right] = F_z[1.2] = .89$$

$p = 0.22$


Normal Probability Plot
0.999
0.997
0.99
0.98
0.95
0.90
0.75
0.50
0.25
0.10
0.05
0.02
0.01
0.003
0.001
Probability
1-p/2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
z
1.5
2
2.5
Data