Introduction to Neural Computation

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MIT BCS 9.40—2018

Lecture 18
Recurrent Neural Networks

Feed-forward neural networks

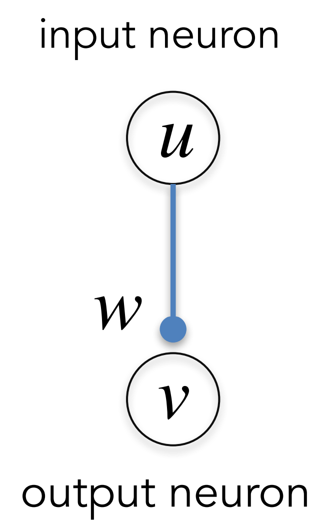
- We have been considering neural networks that use firing rates, rather than spike trains. ('rate model')

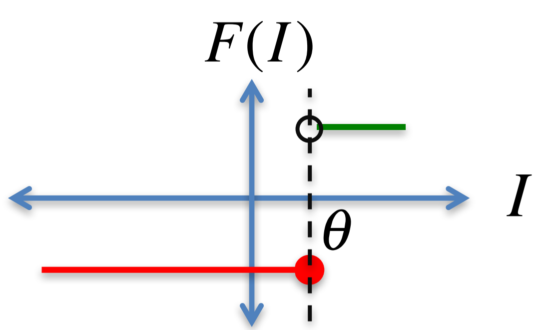
- Synaptic input is the firing rate of the input neuron times a synaptic weight w.

$$I_s = wu$$

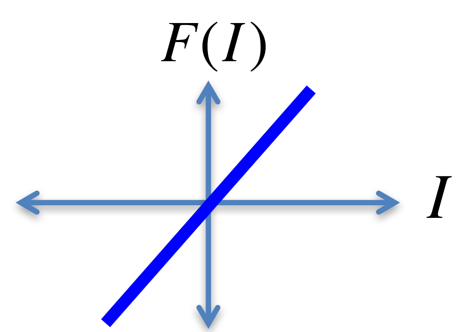
- The output firing rate is some non-linear function of the synaptic input.

$$v = F[I_s] = F[wu]$$





binary threshold neuron

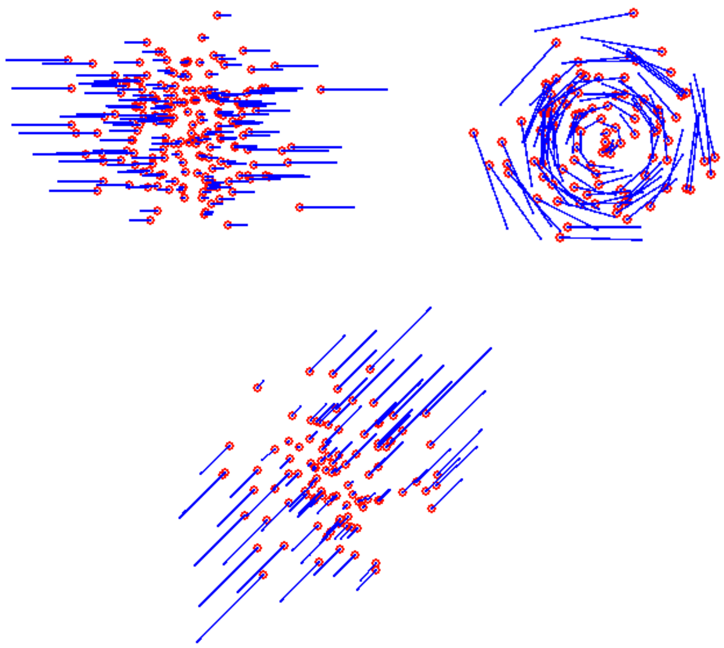


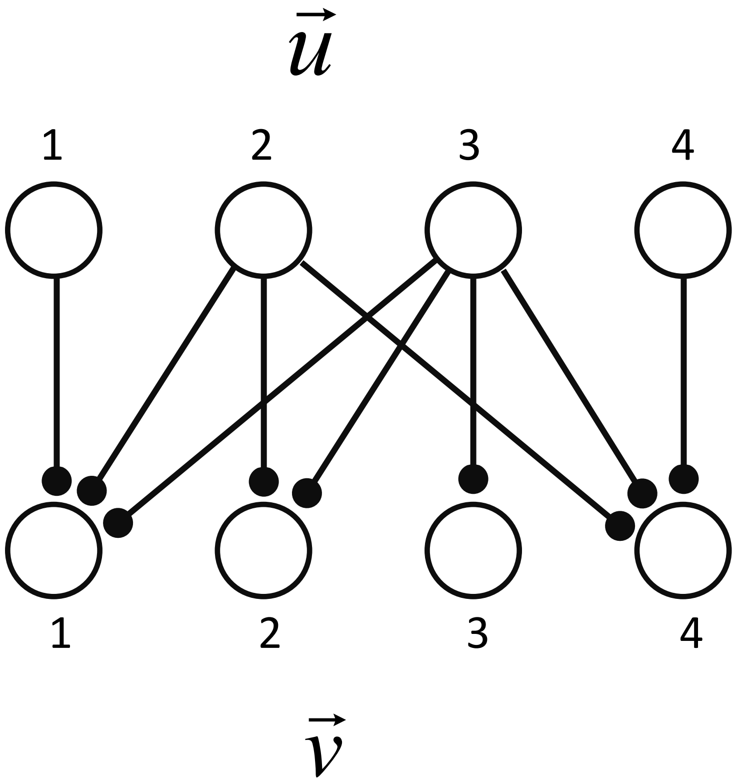
linear neuron $v = wu$

Feed-forward network

- Implements an arbitrary matrix transformation

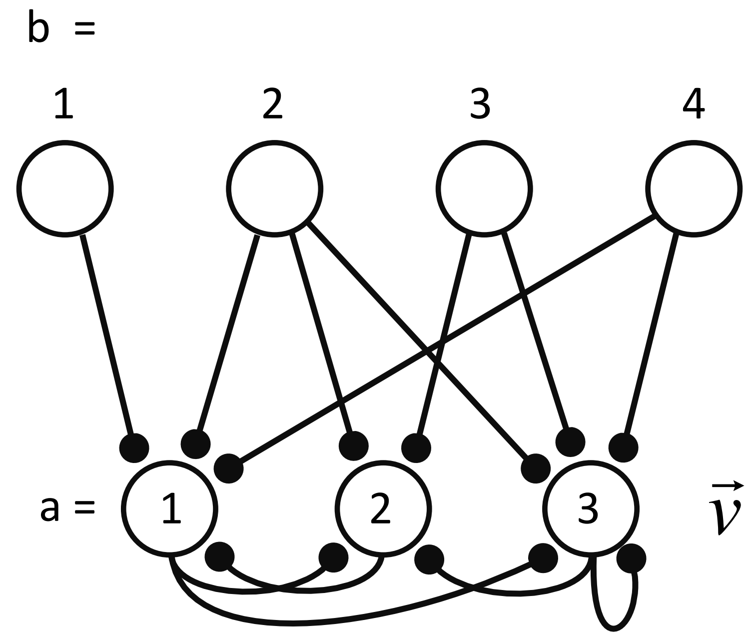
$$\vec{v} = W\,\vec{u}$$





Recurrent networks

- Today we will consider the case where there are also connections between different neurons in the output layer



- Develop an intuition for how recurrent networks respond to their inputs

- Examine computations performed by recurrent networks (amplifier, integrator, sequence generation, short term memory

- Use all the powerful linear algebra tools we have developed!

Learning Objectives for Lecture 18

- Mathematical description of recurrent networks
- Dynamics in simple autapse networks
- Dynamics in fully recurrent networks
- Recurrent networks for storing memories
- Recurrent networks for decision making (winner-take-all)

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Time dependence

- The steady state firing rate of our output neuron looks like this…

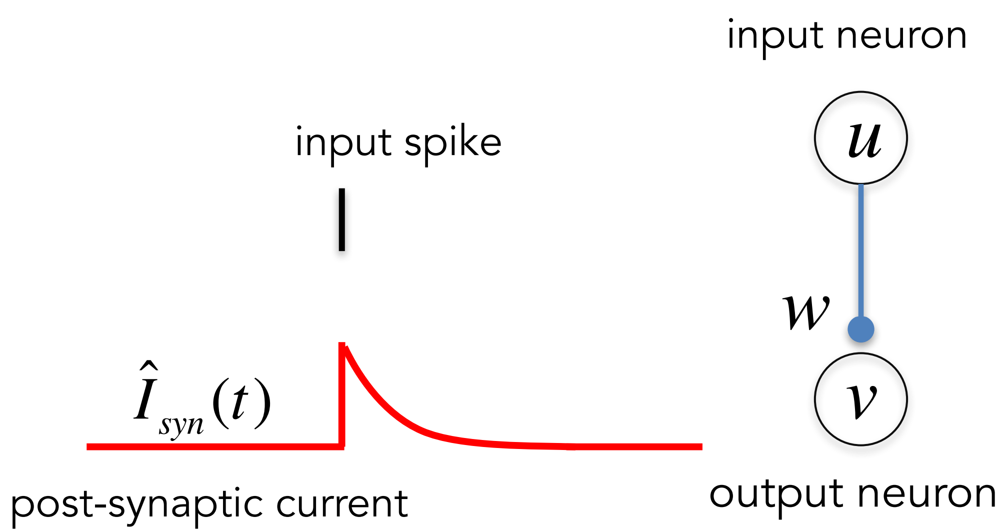
$$v_\infty = F\left[I_s\right] = F\left[wu\right]$$

- But neurons don't respond instantaneously to current inputs

Synaptic delays

Dendritic propagation

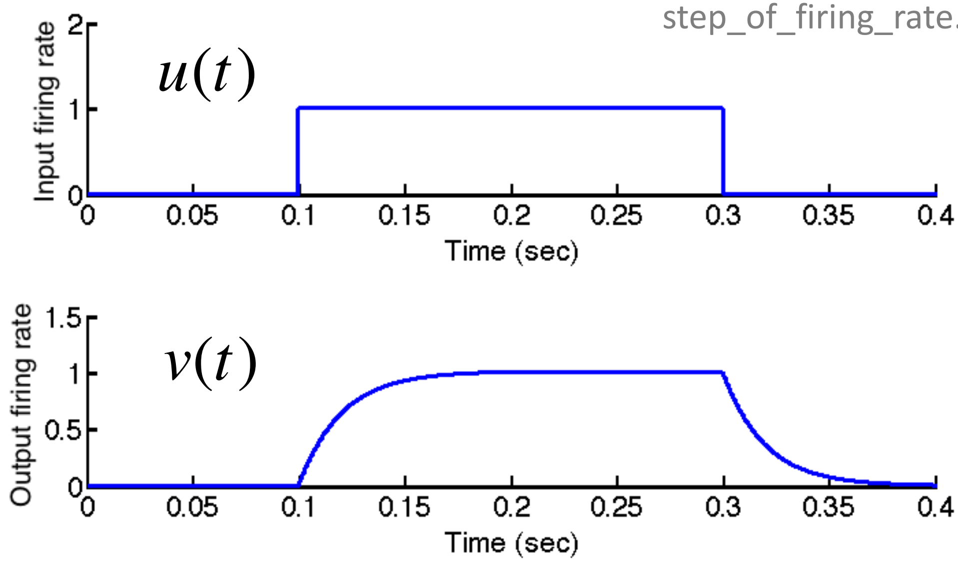
Membrane time constant



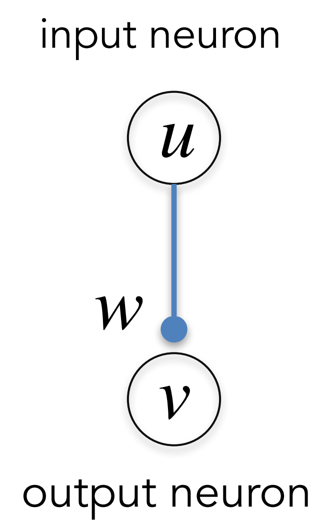
Time dependence

- We model the firing rate of our model neuron as follows:

step_of_firing_rate.m



$$\tau_n \frac{dv}{dt} = -v + v_\infty$$



- We will look at how networks respond to changes in their inputs

$$v_\infty(t) = F\left[wu(t)\right]$$

$$\boxed{\tau_n \frac{dv}{dt} = -v + F\left[wu(t)\right]}$$

Time dependence

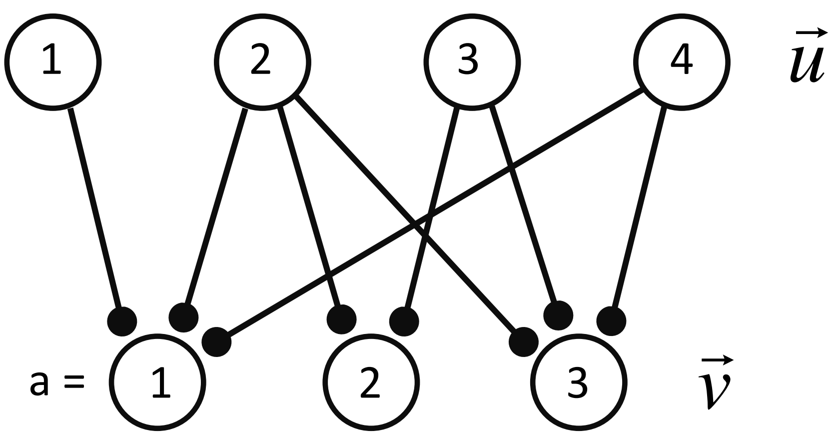
- We can incorporate time-dependence into our general feed-forward network…

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + \vec{v}_\infty$$

$$\vec{v}_\infty = F\left[W\,\vec{u}\right]$$

$$\boxed{\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + F\left[W\,\vec{u}\right]}$$

Feed-forward weight matrix W_{ab}



- The time dependence is really boring in a feedforward network, but it is extremely important in RNNs.

Recurrent networks

- We will now consider the case where there are connections between different neurons in the output layer

- Two kinds of input

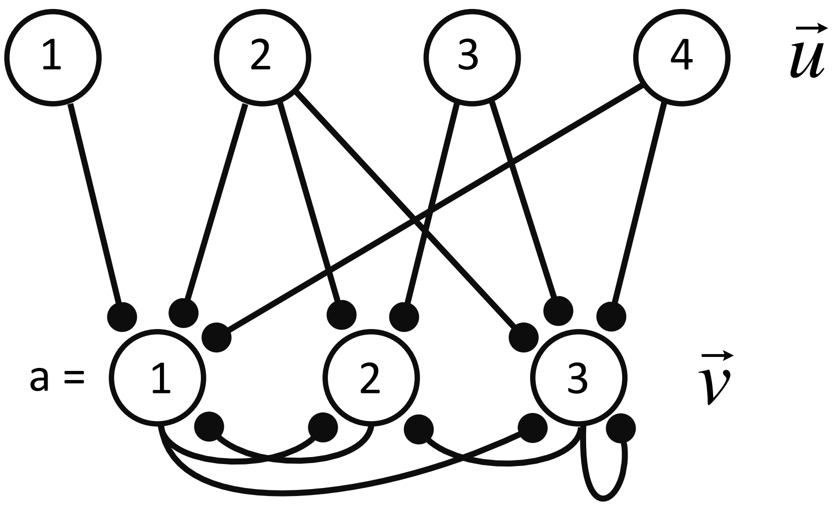
  Feed-forward input

  $$W\,\vec{u}$$

  Recurrent input

  $$M\,\vec{v}$$

Feed-forward weight matrix W_{ab}



Recurrent weight matrix $M_{aa'}$

to **a** from **a'**

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + F\left[W\,\vec{u} + M\,\vec{v}\right]$$

Recurrent networks

- We will now consider the case where there are connections between different neurons in the output layer

Feed-forward input

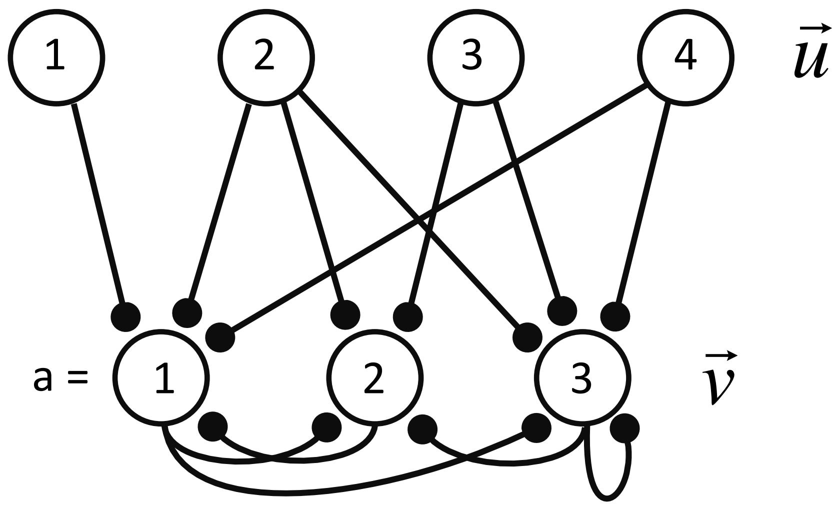
$$W\,\vec{u} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Recurrent input

$$M\,\vec{v} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\boxed{\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + F\left[W\,\vec{u} + M\,\vec{v}\right]}$$

Feed-forward weight matrix W_{ab}



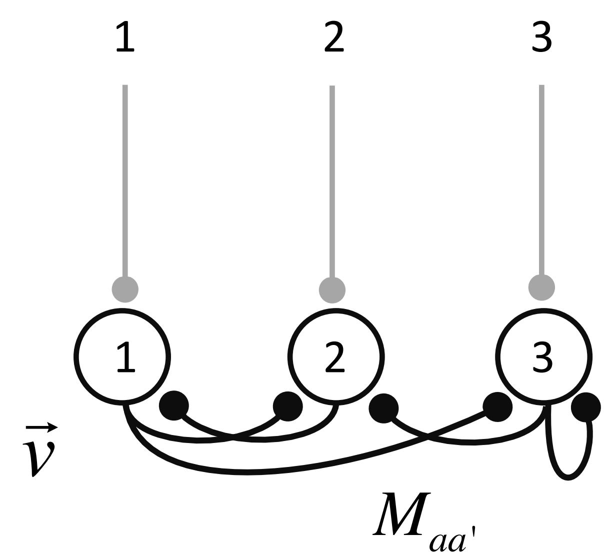
Recurrent weight matrix $M_{aa'}$

to **a** from **a'**

Recurrent networks

- We will simplify this equation to focus on the recurrent network
- Rather than writing the input as a vector of input firing rates, write a vector of effective inputs to each output neuron.

$$\vec{h} = W\,\vec{u}$$

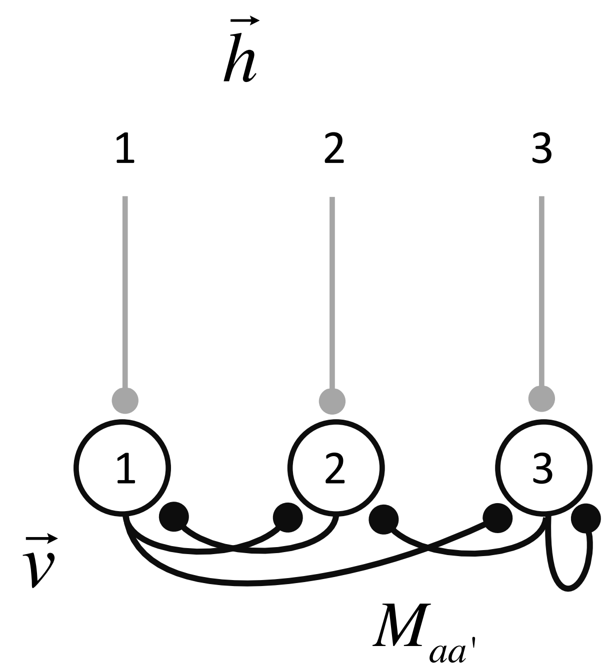


$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + F\left[\vec{h} + M\,\vec{v}\right]$$

Recurrent networks

- We will start by analyzing the case with linear neurons

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + F\left[\vec{h} + M\,\vec{v}\right]$$



- For linear neurons

$$F(\vec{x}) = \vec{x}$$

Thus…

$$\boxed{\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + M\,\vec{v} + \vec{h}}$$

This is a system of coupled equations!

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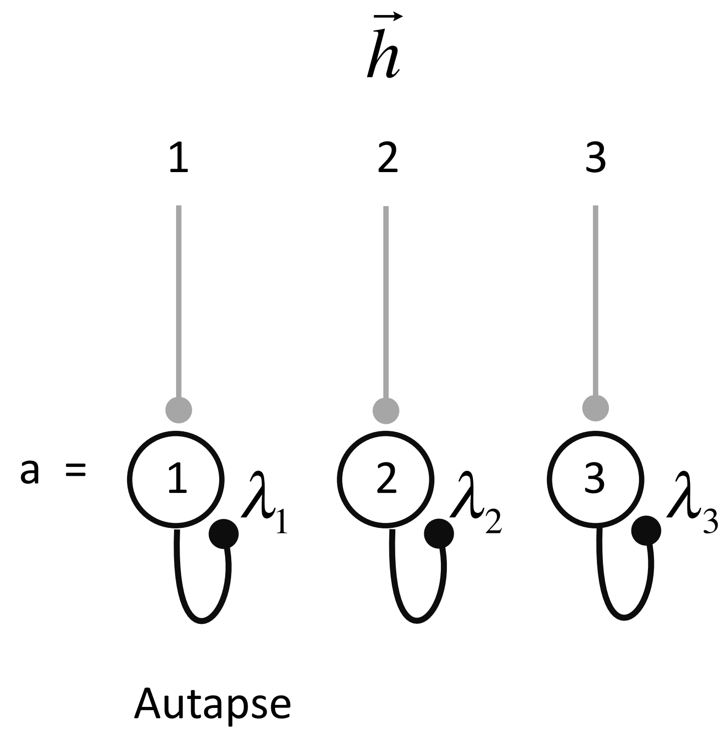
Recurrent networks

- Consider the case that M is a diagonal matrix

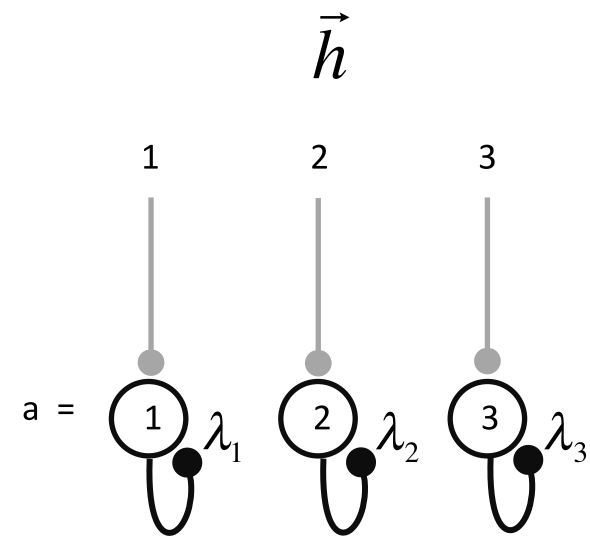
$$M = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & \mathbf{0} & \\ & \mathbf{0} & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$



Recurrent networks

- Note that if M is a diagonal matrix

$$M = \Lambda = \begin{pmatrix} \lambda_1 & & 0 & \\ & \lambda_2 & & \\ 0 & & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$

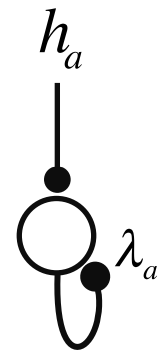


$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + \Lambda \vec{v} + \vec{h}$$

$$\tau_n \frac{dv_a}{dt} = -v_a + \lambda_a v_a + h_a$$

We have n independent equations – each neuron acts independently of all the others

Recurrent networks



- Rewrite our equation:

$$\tau_n \frac{dv_a}{dt} = -v_a + \lambda_a v_a + h_a$$

- There are three cases to consider

$$\tau_n \frac{dv_a}{dt} = -\underbrace{(1-\lambda_a)}_{>0 \quad =0 \quad <0} v_a + h_a$$

- Start with the case that: $\lambda_a < 1$

$$\underbrace{\frac{\tau_n}{1-\lambda_a}} \frac{dv_a}{dt} = -v_a + \underbrace{\frac{h_a}{1-\lambda_a}}$$

$$\tau_a \frac{dv_a}{dt} = -v_a + v_{a,\infty}$$

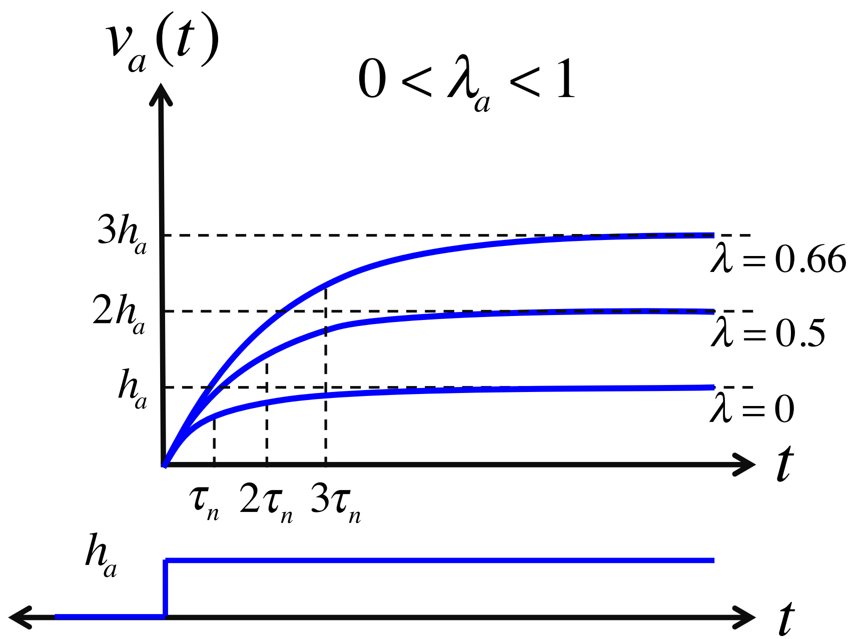
A solution we've seen before!

$$v_a(t) = v_{a,\infty} + (v_0 - v_{a,\infty})e^{-t/\tau_a}$$

Exponential relaxation

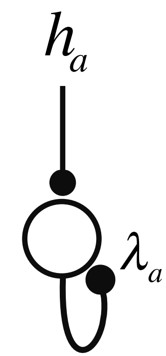
Recurrent networks

- Positive (excitatory) feedback acts to amplify the steady state activity of each neuron by an amount that depends on the strength of the feedback!



$$v_{a,\infty} = \frac{h_a}{1-\lambda_a}$$

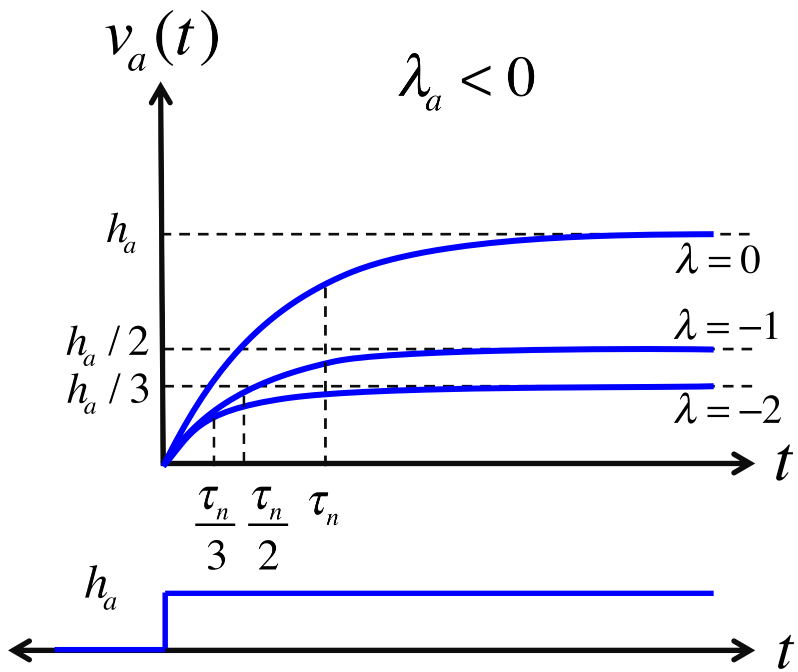
$$\tau_a = \frac{\tau_n}{1-\lambda_a}$$



- Positive feedback amplifies the response and slows the time-constant of the response

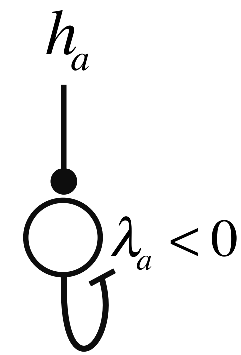
Recurrent networks

- Negative (inhibitory) feedback acts to suppress the steady state activity of a neuron by an amount that depends on the strength of the feedback.



$$v_{a,\infty} = \frac{h_a}{1-\lambda_a}$$

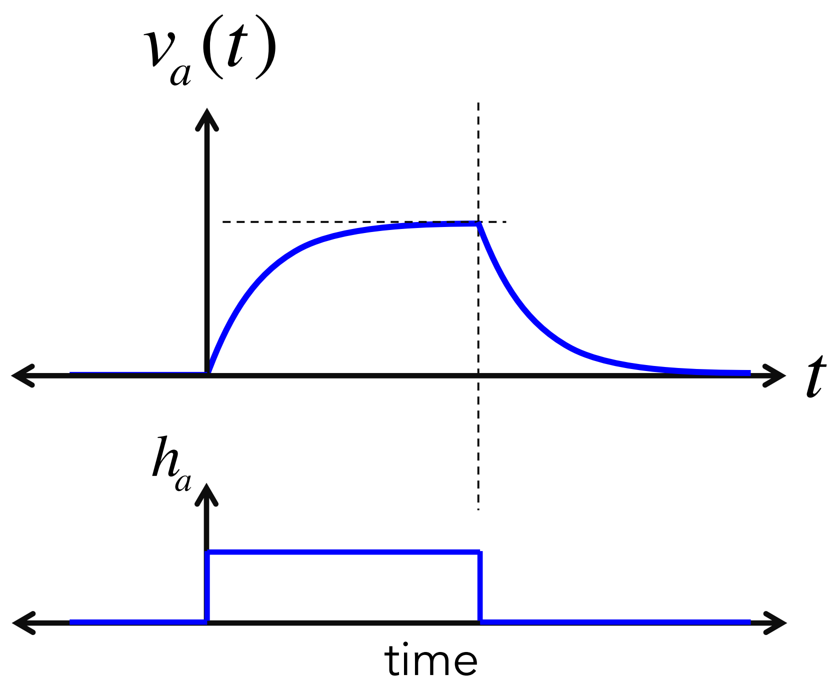
$$\tau_a = \frac{\tau_n}{1-\lambda_a}$$

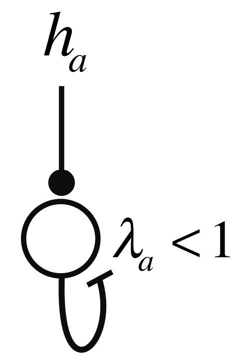


- Negative feedback suppresses the response and speeds the time-constant of the response

Recurrent networks

- If $\lambda < 1$, the activity always relaxes back to zero when the input is removed.

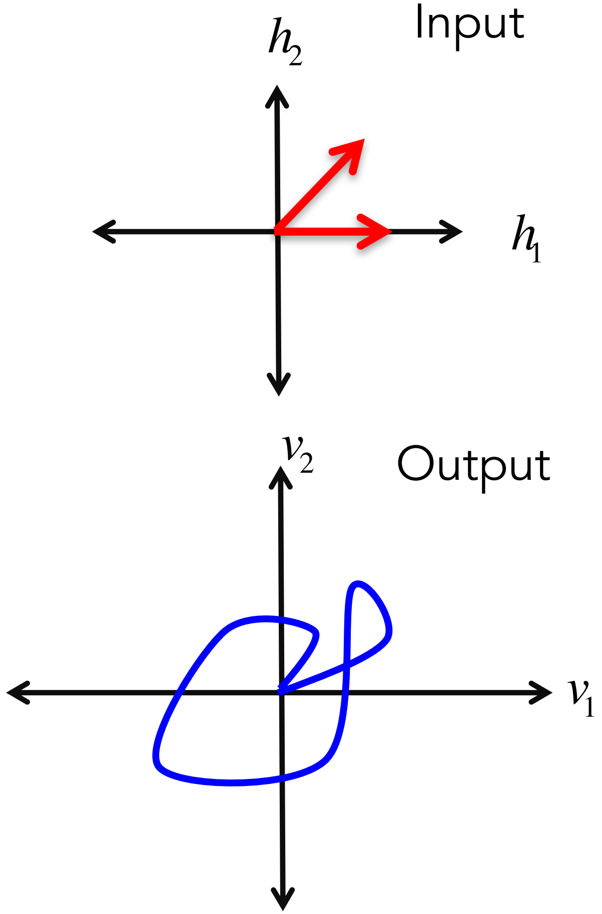


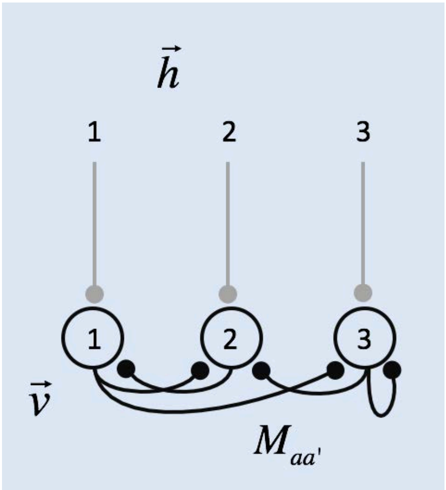


Recurrent networks

- How do we represent the response of a network of neurons.

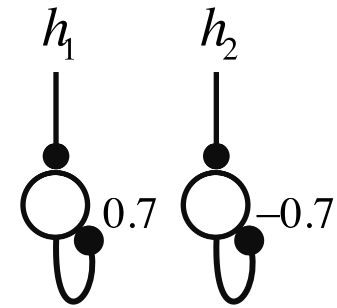
State-space trajectories

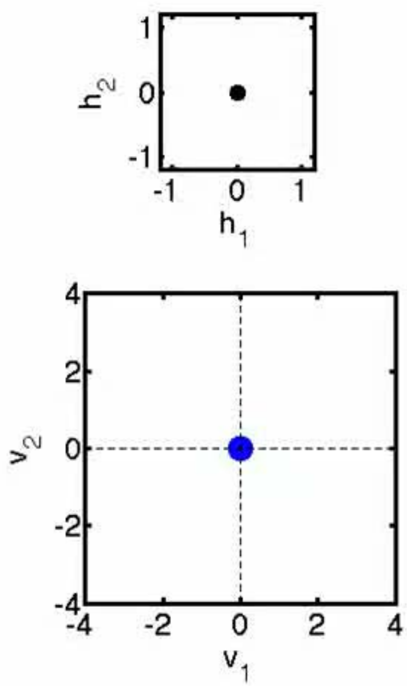


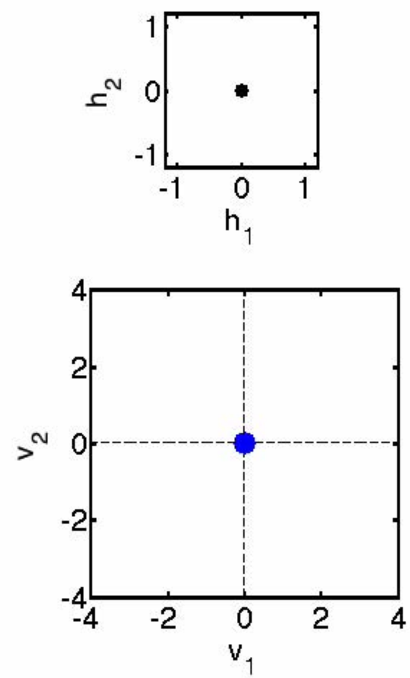


- Cool computations
  - Amplifier
  - Integrator
  - Memory
  - Sequence generator

State-space trajectories







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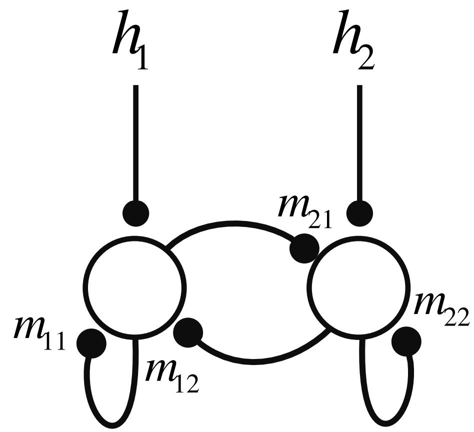
Recurrent networks

- Now let's look at the more general case of recurrent connectivity.



$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

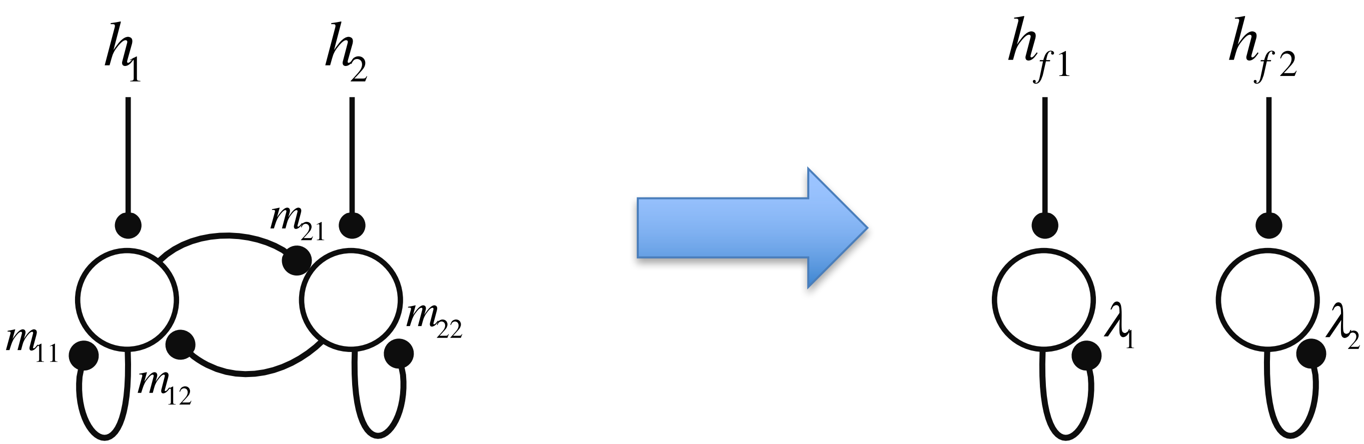
Recurrent networks

- We saw how the behavior of a recurrent network is extremely simple to describe if M is diagonal.

- So let's make M diagonal! Rewrite M as follows

$$M = \Phi \Lambda \Phi^T$$

where Λ is a diagonal matrix.



Recurrent networks

- How do we write M as $\Phi\Lambda\Phi^T$?
- Solve the eigenvalue equation $M\Phi = \Phi\Lambda$
  - The diagonal elements of Λ are the eigenvalues of M

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 & \\ & \lambda_2 & & \\ 0 & & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$

  - The columns of Φ are the eigenvectors of M

$$\Phi = \left[\hat{f}_1 \middle| \hat{f}_2 \middle| \hat{f}_3 \quad \cdots \middle| \hat{f}_n \right]$$

- Remember that…

$$M\,\hat{f}_\alpha = \lambda_\alpha\,\hat{f}_\alpha$$

Recurrent networks

$$M\Phi = \Phi\Lambda \qquad\qquad M\,\hat{f}_\mu = \lambda_\mu\,\hat{f}_\mu$$

- If M is a symmetric matrix, then …
    - the eigenvalues are real
    - Φ is a rotation matrix. The eigenvectors give us an orthogonal basis set:

$$\hat{f}_i \cdot \hat{f}_j = \delta_{ij}$$

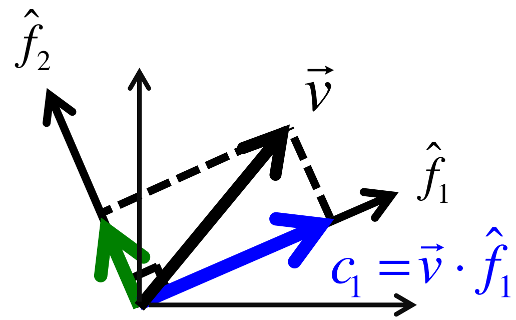
$$\Phi^T\Phi = I$$

Recurrent networks

- Now we are going to write our vector of output firing rates $\vec{v}$ in this new basis.

Project $\vec{v}$ onto each of the new basis vectors.

$$c_\alpha = \vec{v} \cdot \hat{f}_\alpha$$



- Express $\vec{v}$ as a linear combination of basis vectors

$$\vec{v} = c_1 \hat{f}_1 + c_2 \hat{f}_2 + c_2 \hat{f}_3 + \ldots$$

Recurrent networks

- Of course $\vec{v}$ is a function of time, so we have to write…

$$\vec{v}(t) = c_1(t)\hat{f}_1 + c_2(t)\hat{f}_2 + c_3(t)\hat{f}_3 + \ldots$$

or

$$\vec{v}(t) = \sum_{i=1}^{n} c_i(t)\hat{f}_i$$

where

$$c_\alpha(t) = \vec{v}(t)\cdot\hat{f}_\alpha$$

- In matrix notation, we write this change-of-basis as

$$\vec{v} = \Phi\vec{c} \qquad\qquad \vec{c} = \Phi^T\vec{v}$$

Recurrent networks

- Let's rewrite our network equation in this new basis set…

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + M\vec{v} + \vec{h} \qquad \vec{v} = \Phi\vec{c}$$

$$\tau_n \Phi \frac{d\vec{c}}{dt} = -\Phi\vec{c} + \boxed{M\Phi}\,\vec{c} + \vec{h}$$

- But we have chosen a basis set Φ such that

$$M\Phi = \Phi\Lambda$$

- Thus…

$$\tau_n \Phi \frac{d\vec{c}}{dt} = -\Phi\vec{c} + \Phi\Lambda\vec{c} + \vec{h}$$

Recurrent networks

$$\tau_n \Phi \frac{d\vec{c}}{dt} = -\Phi\vec{c} + \Phi\Lambda\vec{c} + \vec{h}$$

- Multiply both sides from the left by Φ^T

$$\tau_n \underbrace{\Phi^T \Phi} \frac{d\vec{c}}{dt} = -\underbrace{\Phi^T \Phi}\vec{c} + \underbrace{\Phi^T \Phi}\Lambda\vec{c} + \Phi^T \vec{h}$$

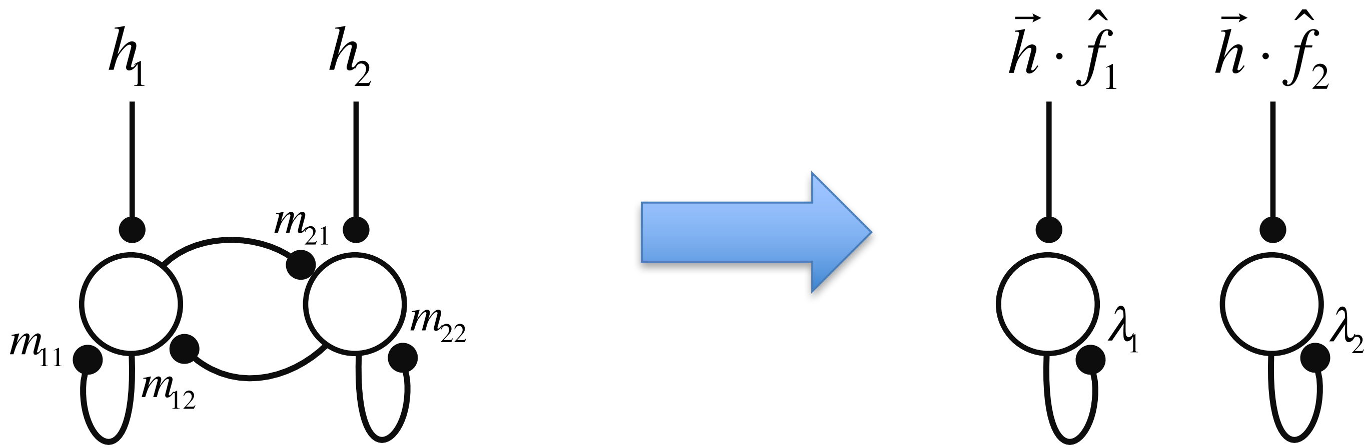
$$\tau_n \frac{d\vec{c}}{dt} = -\vec{c} + \Lambda\vec{c} + \vec{h}_f \qquad \vec{h}_f = \Phi^T \vec{h}$$

- But this is just our original network equation with a diagonal weight matrix!

Recurrent networks

- We can rewrite the equation for our network as n independent equations for n independent 'modes' of the network

- We can think of this transformation as making a new network with only autapses.



- The activities $c_\alpha(t)$ of our network modes represent activity of linear combinations of neurons in our original network

Recurrent networks

- Let's find the steady-state solution of our system of equations…

$$\tau_n \frac{d\vec{c}}{dt} = -\vec{c} + \Lambda\vec{c} + \Phi^T\vec{h}$$

$$\tau_n \frac{d\vec{v}}{dt} = -\vec{v} + M\vec{v} + \vec{h}$$

$$\tau_n \frac{d\vec{c}}{dt} = -I\vec{c} + \Lambda\vec{c} + \Phi^T\vec{h}$$

$$\vec{v} = \Phi\vec{c}$$

$$\cancel{\tau_n \frac{d\vec{c}}{dt}} = -(I-\Lambda)\vec{c} + \Phi^T\vec{h}$$

$$0 = -(I-\Lambda)\vec{c}_\infty + \Phi^T\vec{h}$$

$$(I-\Lambda)\vec{c}_\infty = \Phi^T\vec{h}$$

$$\vec{c}_\infty = (I-\Lambda)^{-1}\Phi^T\vec{h}$$

$$\underbrace{\Phi\vec{c}_\infty} = \Phi(I-\Lambda)^{-1}\Phi^T\vec{h}$$

$$\vec{v}_\infty = \Phi\vec{c}_\infty$$

$$\vec{v}_\infty = \Phi(I-\Lambda)^{-1}\Phi^T\vec{h}$$

Recurrent networks

- The steady-state solution (with input vector $\vec{h}$) is:

$$\vec{v}_\infty = \underbrace{\Phi(I-\Lambda)^{-1}\Phi^T}\ \vec{h} \qquad\qquad \vec{v}_\infty = G\ \vec{h}$$

eigenvectors?

this matrix has the same eigenvectors as M !

and has eigenvalues $g_\mu = \dfrac{1}{1-\lambda_\mu}$ $\qquad G\,\hat{f}_\mu = g_\mu\,\hat{f}_\mu$

- So what happens if our input is parallel to one of the eigenvectors?

$$\vec{h} = \hat{f}_\mu \qquad\qquad \vec{v}_\infty = G\,\hat{f}_\mu \qquad\qquad \vec{v}_\infty = \frac{1}{1-\lambda_\mu}\,\hat{f}_\mu$$

- Then, in steady state, the output will be parallel to the input!

Recurrent networks

- If our input vector is parallel to one of the eigenvectors, then our steady-state output will be parallel to the input.

- In this case, our input activates only one mode of the network, and no other mode.

- The response of the network to inputs along each of the eigenvectors (modes) is amplified or suppressed by a gain factor

$$g_\mu = \frac{1}{1-\lambda_\mu}$$

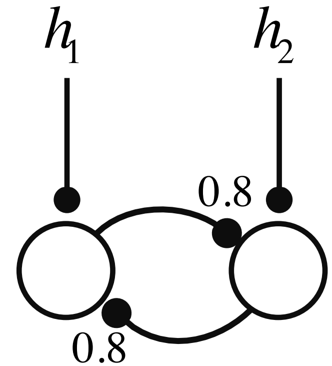
- The time constant of the response is increased or decreased by the same factor

$$\tau_\mu = \frac{\tau_n}{1-\lambda_\mu}$$

Recurrent networks

- Now let's look at a case where two output neurons are connected to each other by mutual excitation.

What is the weight matrix?

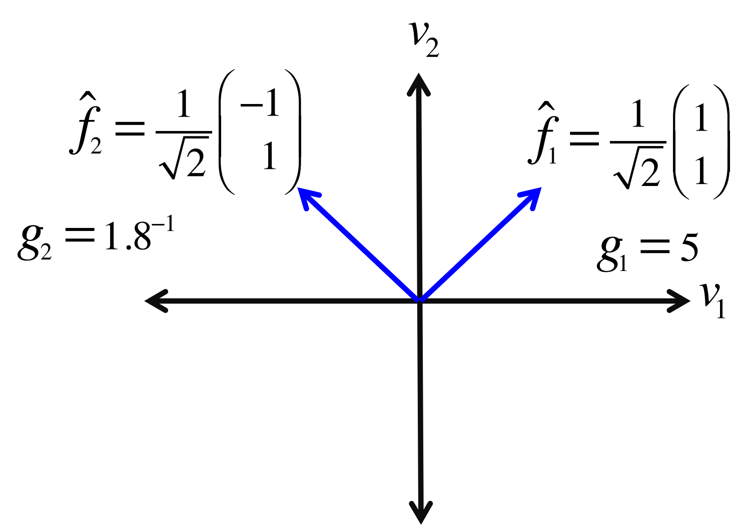


$$M = \begin{pmatrix} 0 & 0.8 \\ 0.8 & 0 \end{pmatrix}$$

$$M\Phi = \Phi\Lambda$$

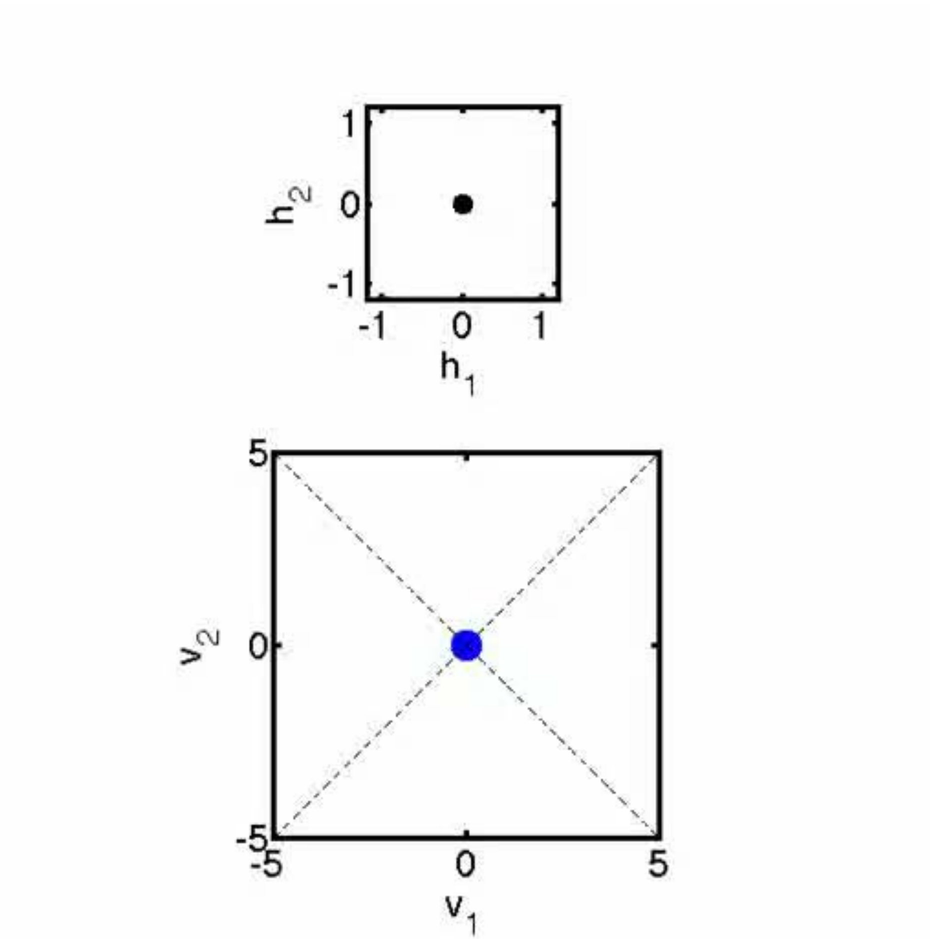
$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

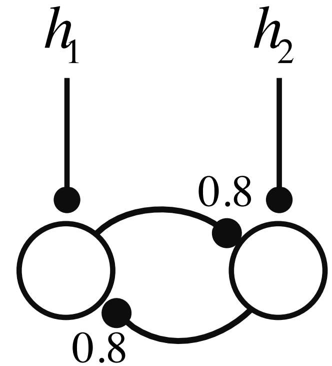
$$\Lambda = \begin{pmatrix} 0.8 & 0 \\ 0 & -0.8 \end{pmatrix}$$

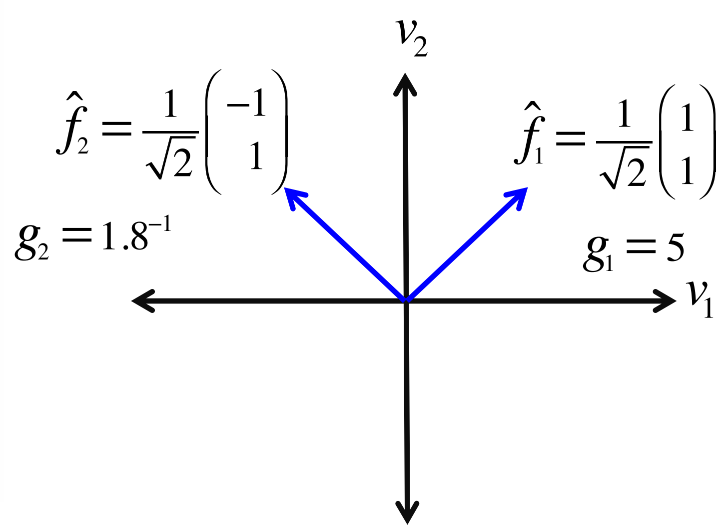


Recurrent networks

- If the input is parallel to the eigenvectors, then only one mode is excited.



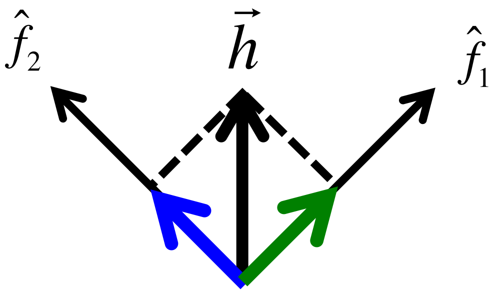


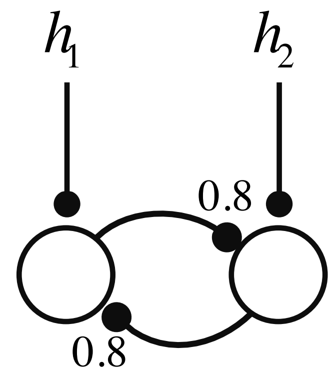


Recurrent networks

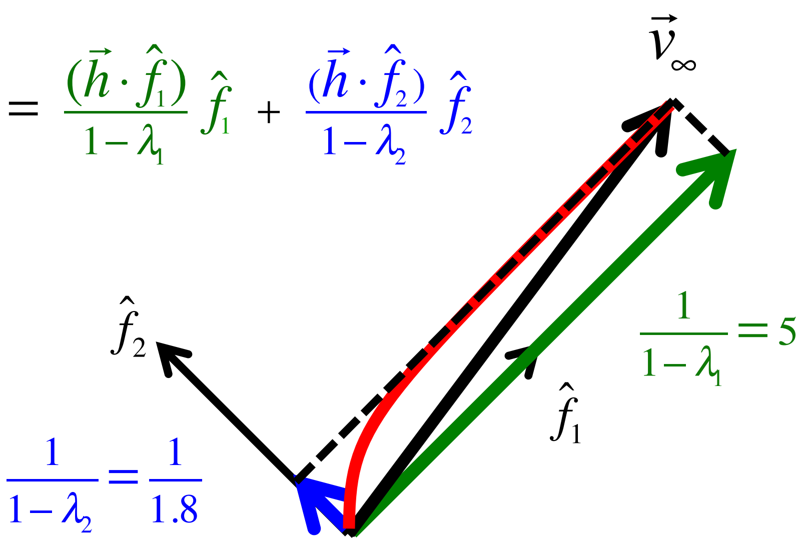
- If the input is not parallel to an eigenvector, we break the input into a component along each mode

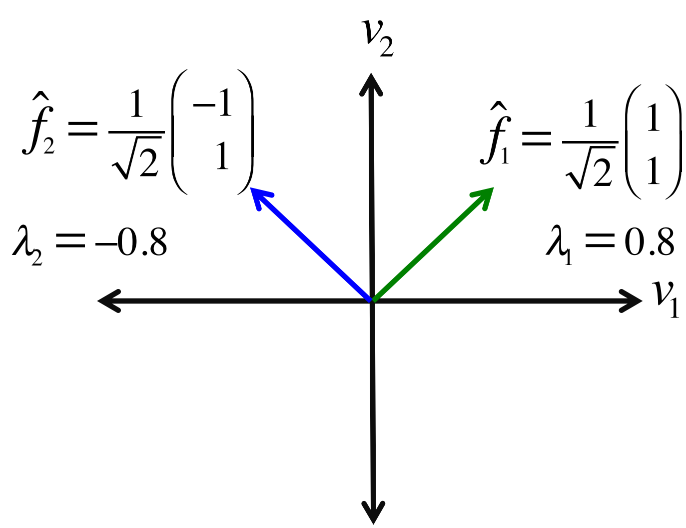
$$\vec{h} = (\vec{h}\cdot\hat{f}_1)\,\hat{f}_1 + (\vec{h}\cdot\hat{f}_2)\,\hat{f}_2$$





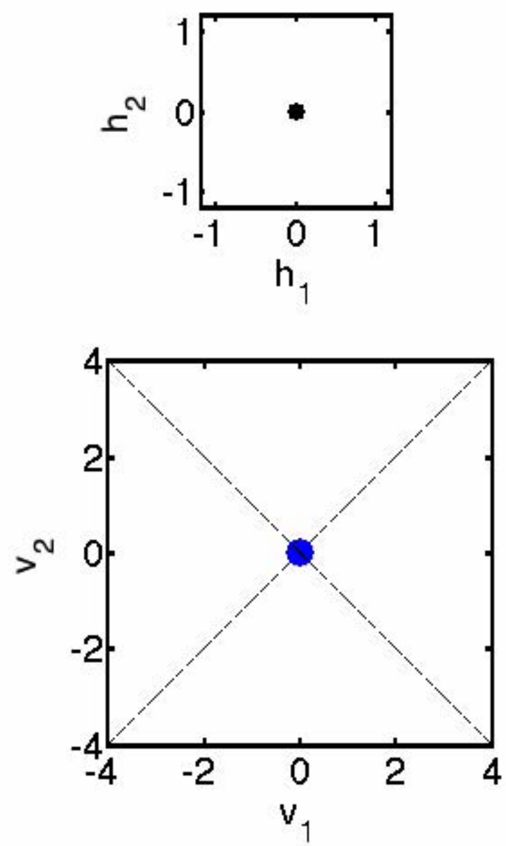
$$\vec{v}_\infty = \frac{(\vec{h}\cdot\hat{f}_1)}{1-\lambda_1}\,\hat{f}_1 + \frac{(\vec{h}\cdot\hat{f}_2)}{1-\lambda_2}\,\hat{f}_2$$

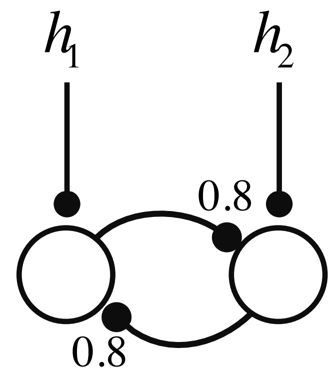


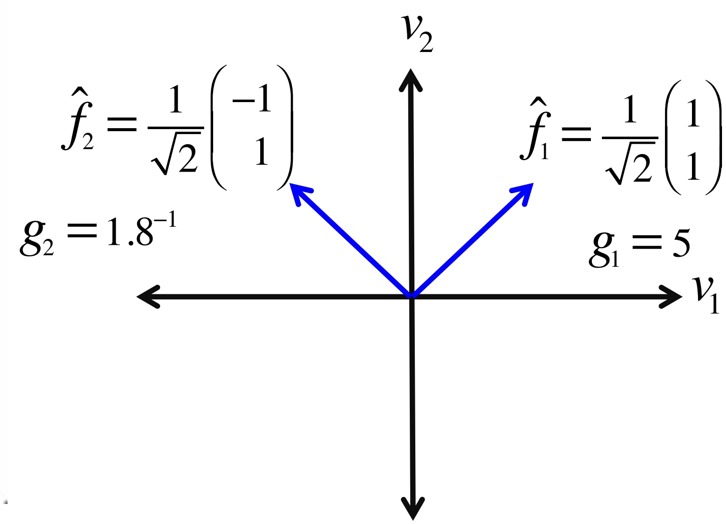


Recurrent networks

- Two output neurons are connected to each other by mutual excitation.

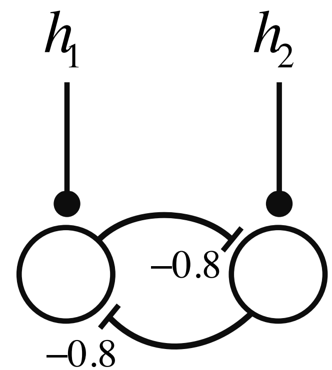






Recurrent networks

- Now let's look at a case where two output neurons are connected to each other by mutual inhibition.



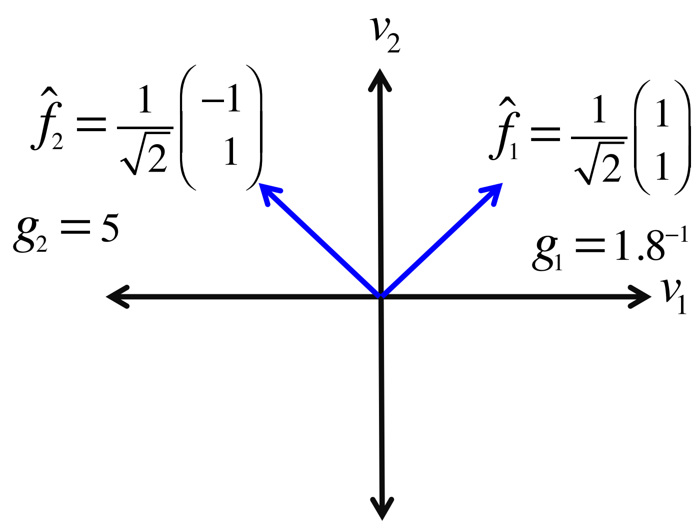
What is the weight matrix?

$$M = \begin{pmatrix} 0 & -0.8 \\ -0.8 & 0 \end{pmatrix}$$

$$M\Phi = \Phi\Lambda$$

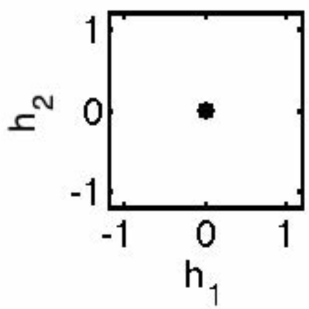
$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

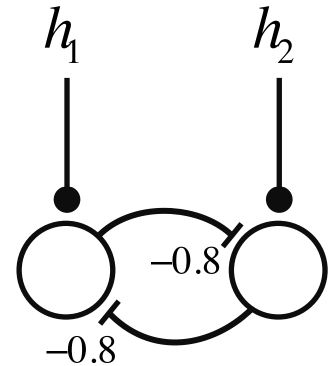
$$\Lambda = \begin{pmatrix} -0.8 & 0 \\ 0 & 0.8 \end{pmatrix}$$

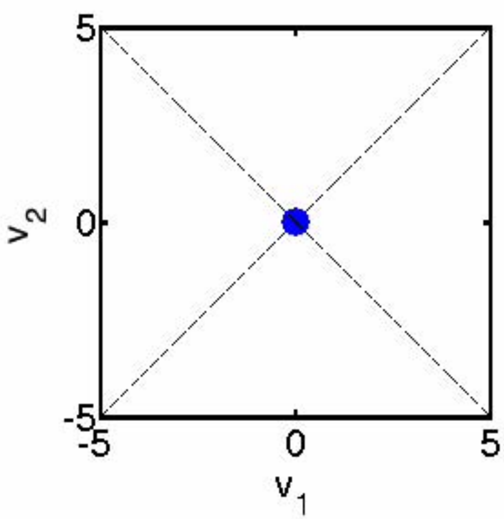


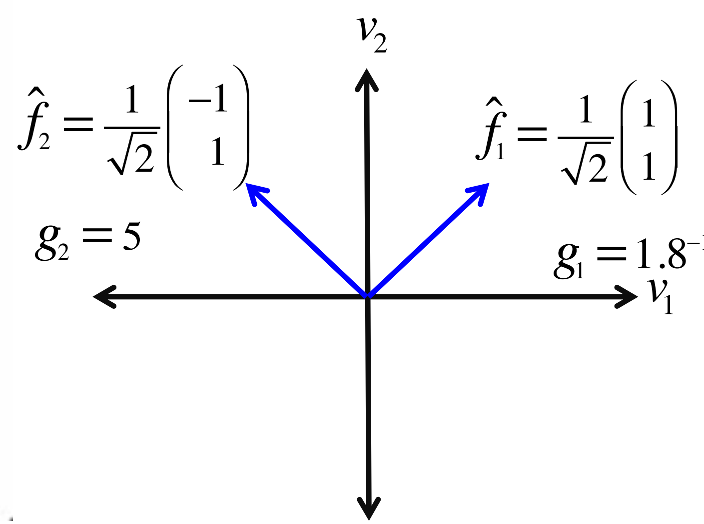
Recurrent networks

- Two output neurons are connected to each other by mutual inhibition.









Learning Objectives for Lecture 18

- Mathematical description of recurrent networks
- Dynamics in simple autapse network
- Dynamics in fully recurrent networks
- Recurrent networks for storing memories
- Recurrent networks for decision making (winner-take-all)

Recurrent networks

- We have described the case where $\lambda < 1$.

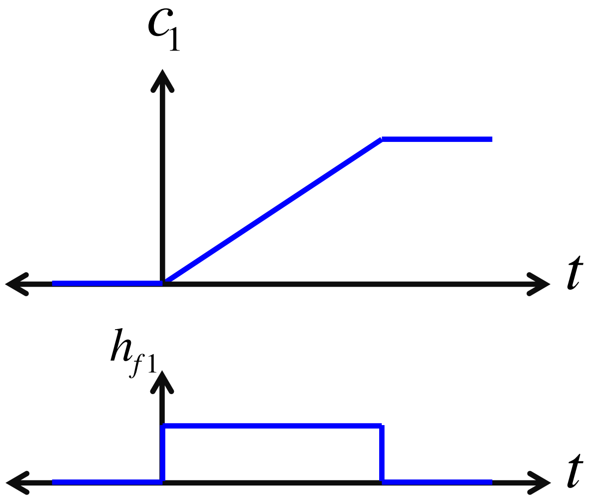
  What happens when $\lambda = 1$?

$$\tau_n \frac{dc_\alpha}{dt} = -\underbrace{(1-\lambda_\alpha)}_{0} c_\alpha + \hat{f}_\alpha \cdot \vec{h}(t)$$

$$\tau_n \frac{dc_1}{dt} = \hat{f}_1 \cdot \vec{h}(t) = h_{f1}(t)$$

$$c_1(t) = c_1(0) + \frac{1}{\tau_n} \int_0^t h_{f1}(\tau)\, d\tau$$

Integrator!



$$h_{f1}(t) = \hat{f}_1 \cdot \vec{h}(t)$$

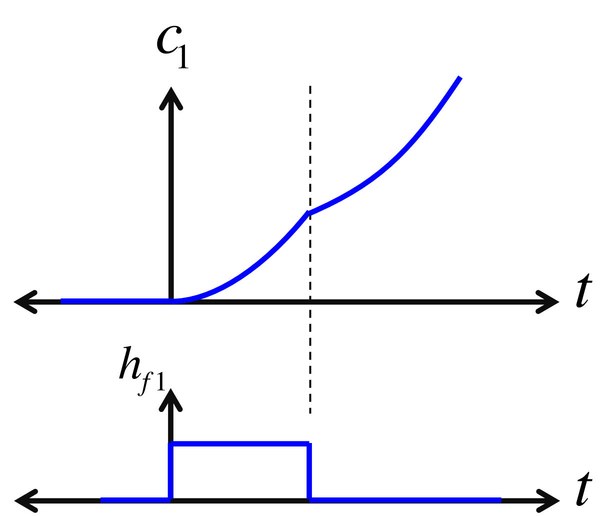
Recurrent networks

- What happens when $\lambda > 1$?

$$\tau_n \frac{dc_1}{dt} = -(1-\lambda_1)c_1 + \hat{f}_1 \cdot \vec{h}(t)$$

$$\tau_n \frac{dc_1}{dt} = \underbrace{(\lambda_1 - 1)}_{>0}c_1 + \hat{f}_1 \cdot \vec{h}(t)$$

Exponential growth!



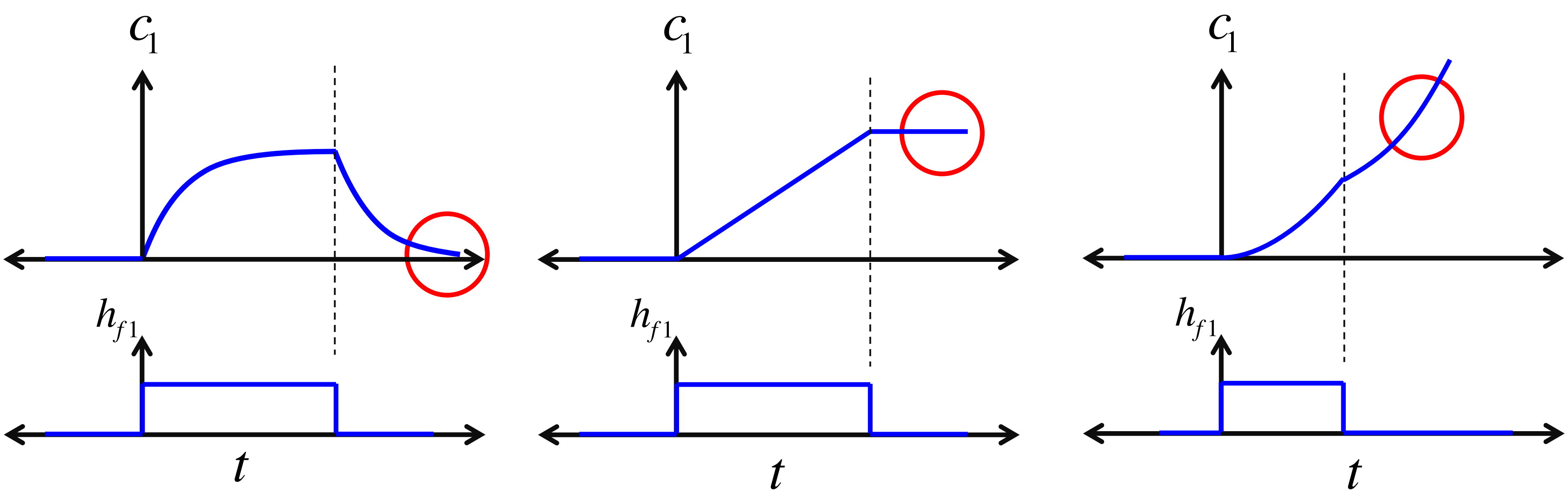
$$h_{f1}(t) = \hat{f}_1 \cdot \vec{h}(t)$$

Recurrent networks

- The behavior of the network depends critically on λ

$\lambda < 1$ | $\lambda = 1$ | $\lambda > 1$

Exponential relaxation | Integration | Exponential growth



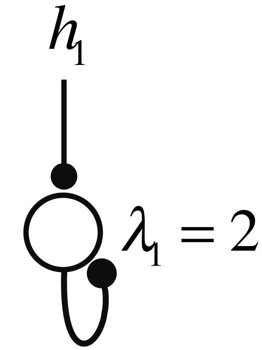
With zero input…
relaxation back to zero

With zero input…
persistent activity!

MEMORY!

Recurrent networks

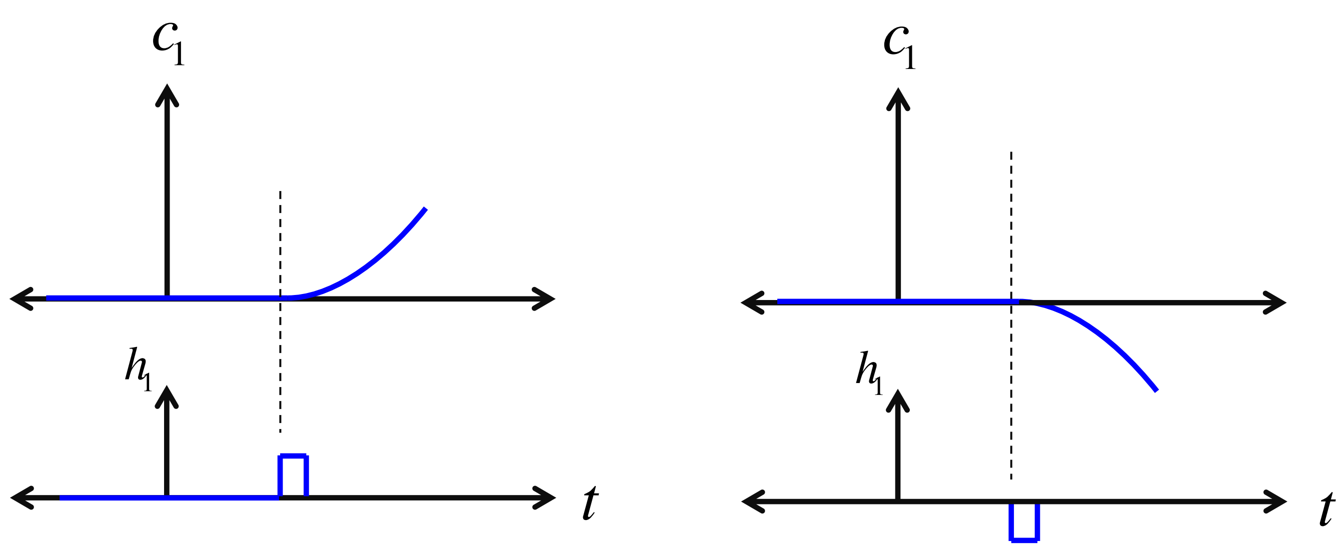
- Networks with $\lambda \geq 1$ have memory!



$$\tau_n \frac{dc_1}{dt} = (\lambda_1 - 1)c_1 + h_{f1}(t)$$

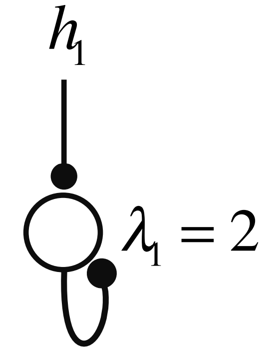
$$\tau_n \frac{dc_1}{dt} = c_1 \qquad\qquad c_1(t) = 0$$

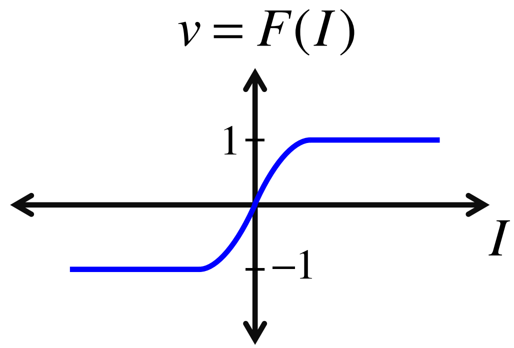
- With zero input, zero is an 'unstable fixed point' of the network

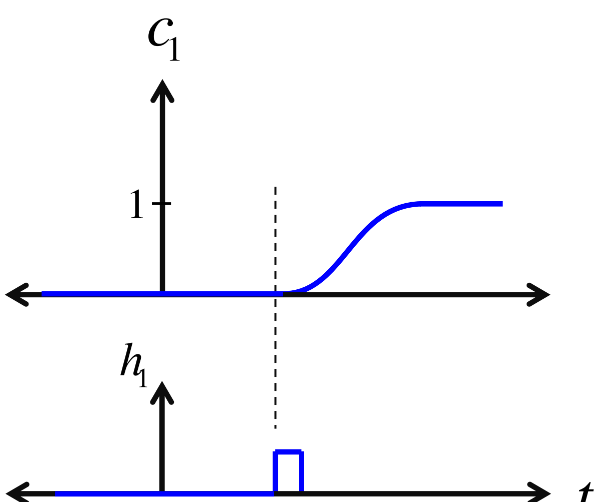


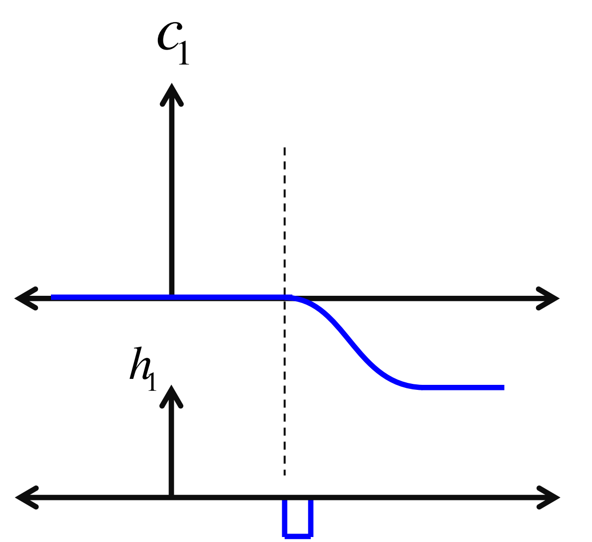
Recurrent networks

- Add a saturating activation function $F(x)$



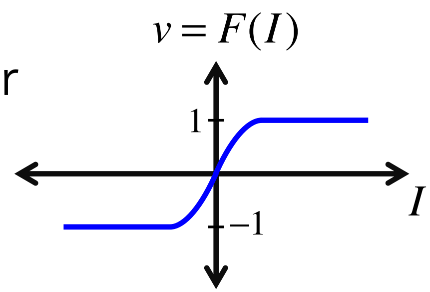


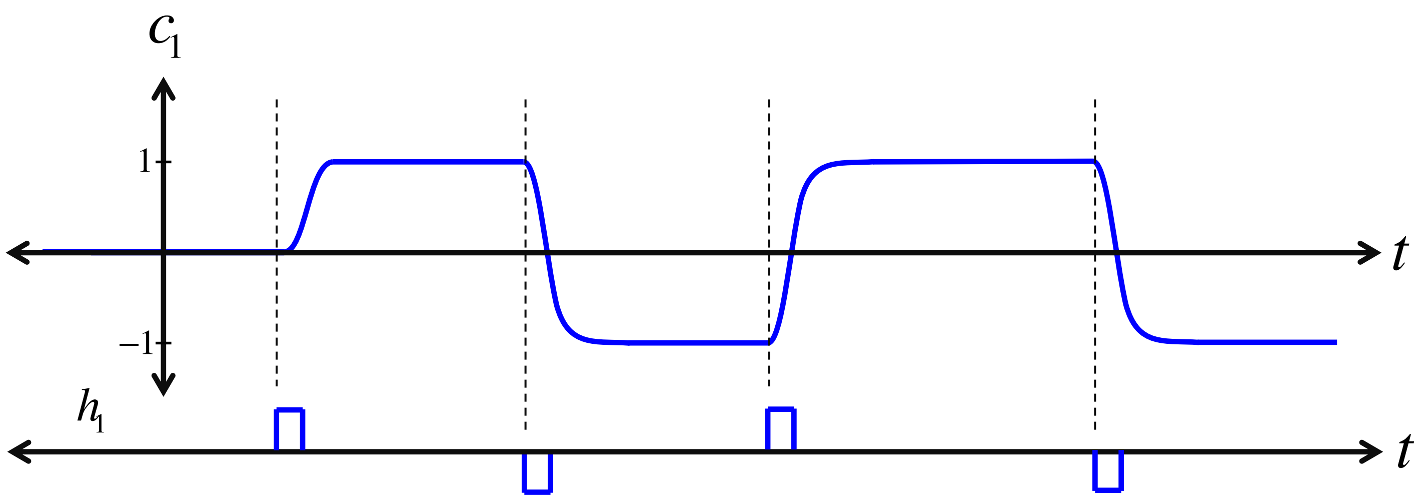


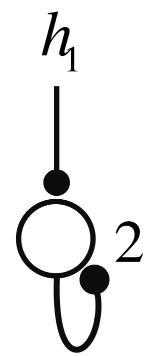


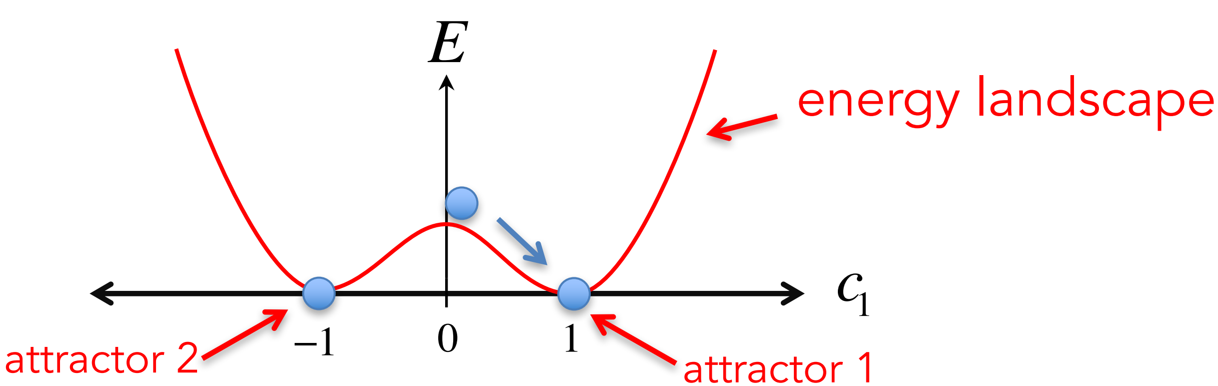
Recurrent networks

- Saturating activation function plus eigenvalues greater than 1 lead to stable states other than zero!



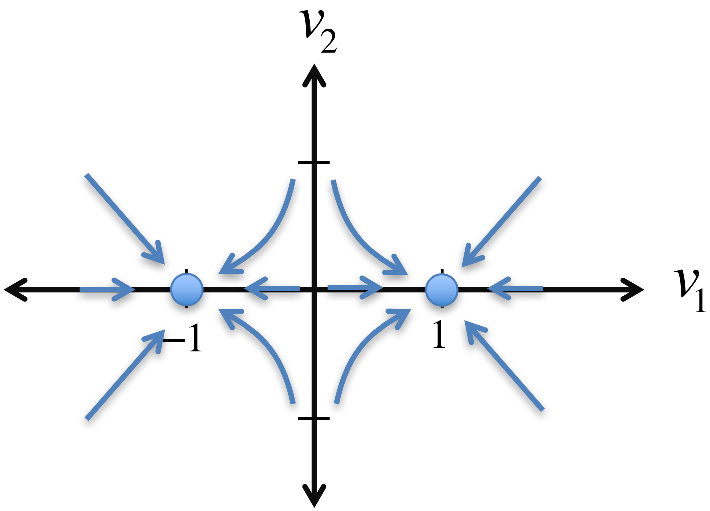


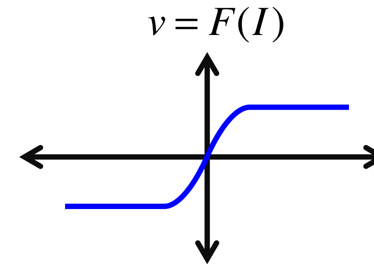


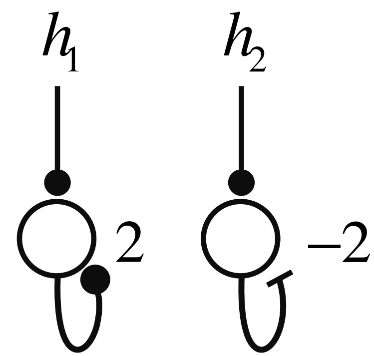


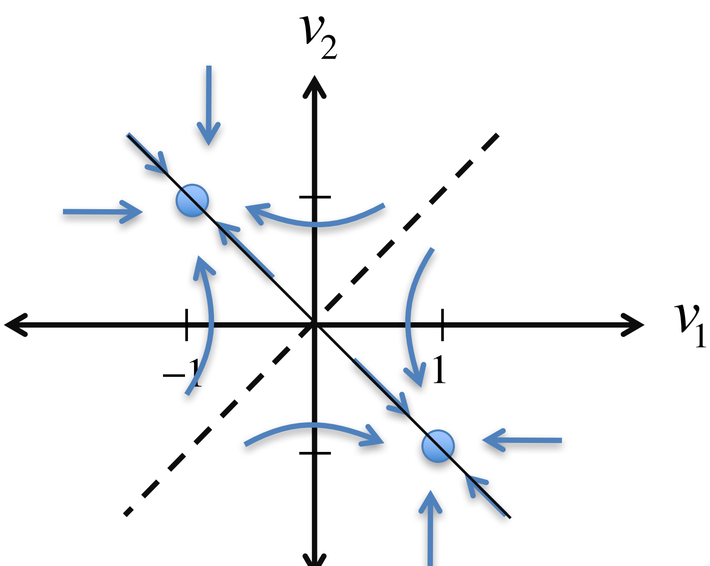
Recurrent networks

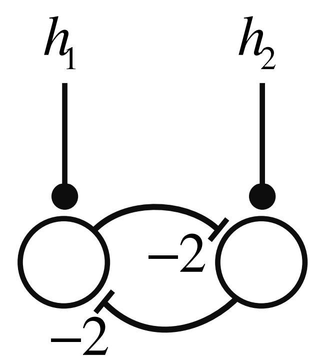
- Two-neuron network that has two attractors











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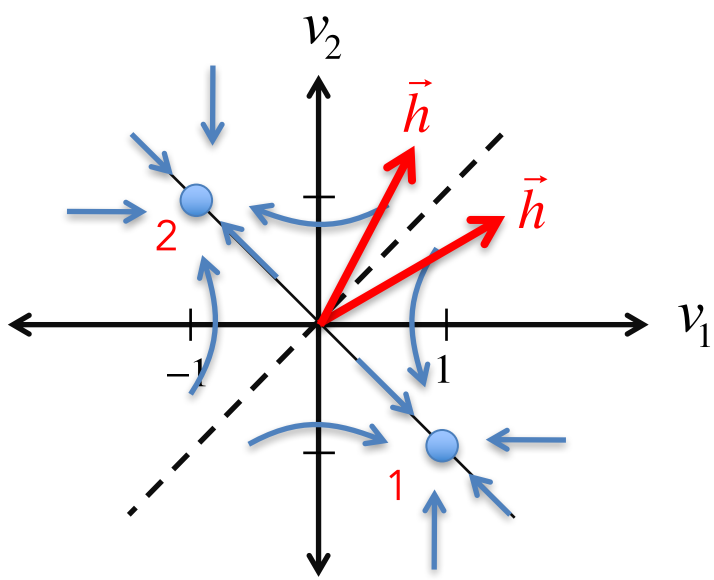
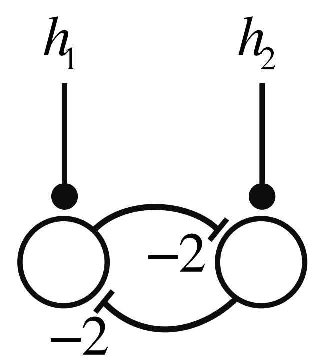
Winner-take-all network

- Implements decision making

  Network will remain in attractor 1 if $h_1 > h_2$

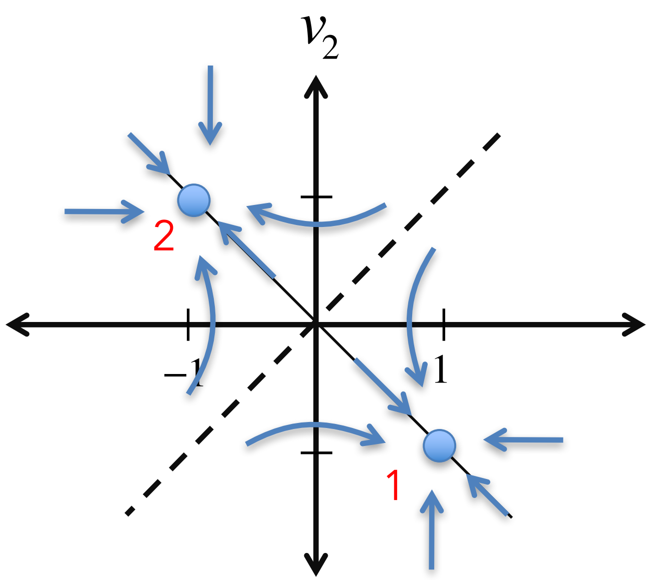
  Network will remain in attractor 2 if $h_2 > h_1$

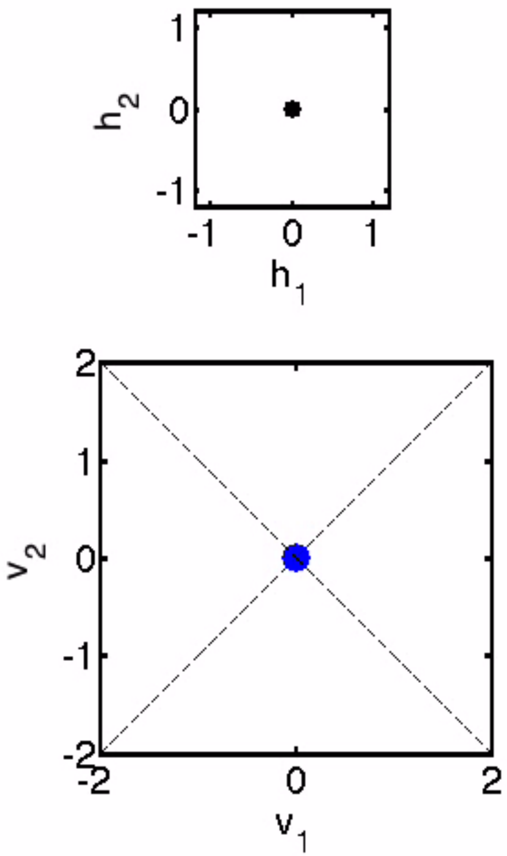




Winner-take-all network

- Implements decision making





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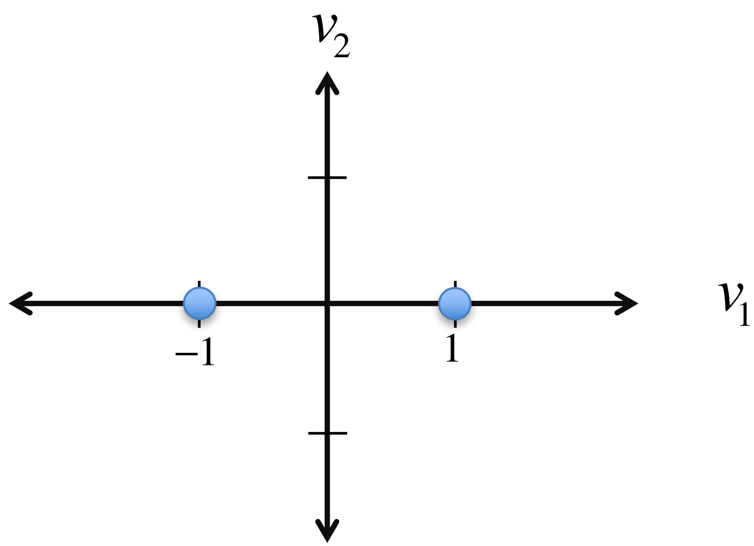
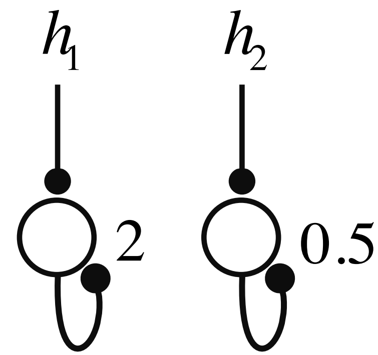
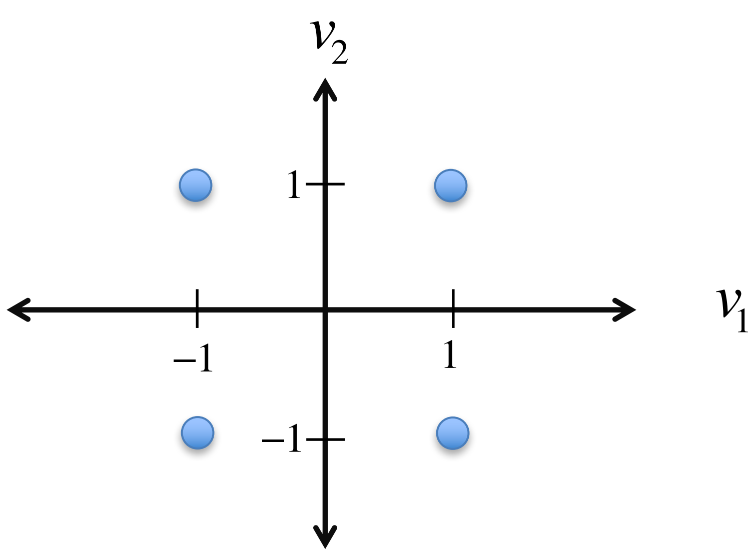
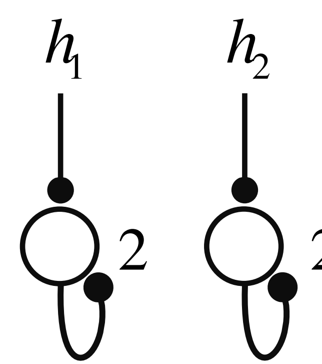
Recurrent networks

- Networks with many attractors…



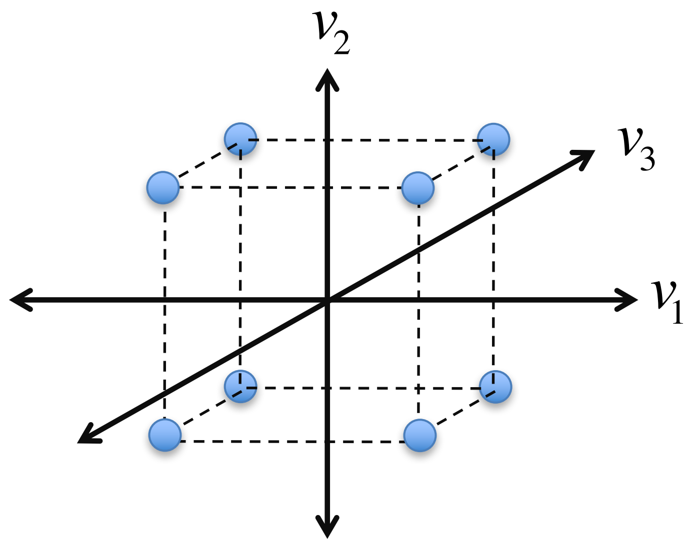


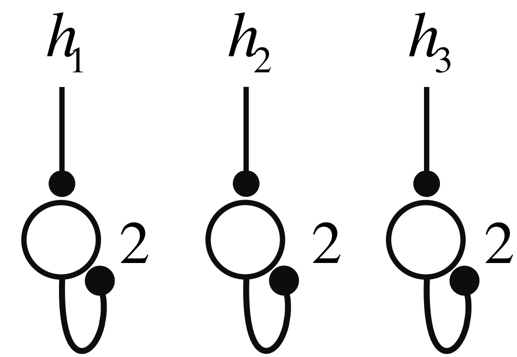




Hopfield networks

- Networks with many attractors…





2^n possible states !