OUTLINE

6.581 / 20.482
LECTURE 21: DECONVOLUTION II

TUESDAY
2 MAY 2006

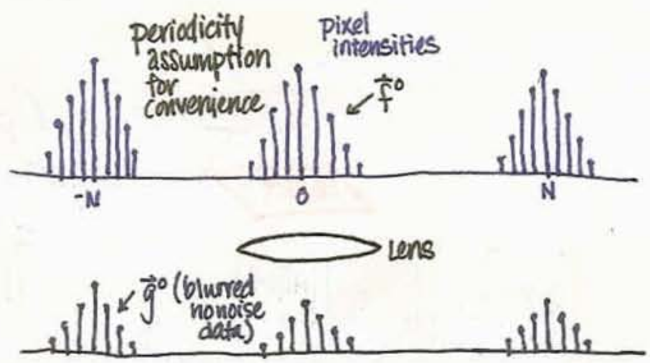
1) Reminder about SVD
2) Fourier (Freq. Domain) View
3) 2D Case

REMINDER

$$\vec{g} = H\vec{f}^{o} + \vec{w}$$

$\vec{g}$: data; H: effects of imaging system; $\vec{f}^{o}$: object; $\vec{w}$: noise

IN 1D



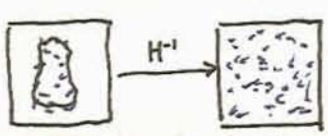
$$\begin{bmatrix} g^o[1] \\ \vdots \\ g^o[N] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[2] & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[3] & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[4] & h[3] \\ \vdots & \vdots & & & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[1] & h[0] \end{bmatrix} \begin{bmatrix} f^o[1] \\ \vdots \\ f^o[N] \end{bmatrix}$$

CIRCULENT MATRIX

1st DECONVOLUTION IDEA

$$\vec{f}^{BE} = H^{-1}\vec{g} = \vec{f}^{o} + H^{-1}\vec{w}$$

(bad estimate)



Using SVD

$$H = U \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} V^T$$

$$U = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{u}_1 & \cdots & \vec{u}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

U, V^T: orthonormal (define coordinate system)

$$\vec{g} = \sum \vec{u}_i \sigma_i (\vec{v}_i^T \vec{f}^o) + \vec{w}$$

$$\vec{f}^{BE} = \underbrace{\sum \frac{1}{\sigma_i} \vec{v}_i (\vec{u}_i^T \vec{g})}_{H^{-1}\vec{g}}$$

$$\vec{f}^{BE} = \sum \vec{v}_i \left[(\vec{v}_i^T \vec{f}^o) + \frac{1}{\sigma_i} \vec{u}_i^T \vec{w} \right]$$

ESTIMATER

$$\vec{f}^{SE} = \alpha_1 (\vec{v}_1^T \vec{f}^o + \tfrac{1}{\sigma_1} \vec{u}_1^T \vec{w}) \vec{v}_1 + \alpha_2 (\underbrace{\vec{v}_2^T \vec{f}^o}_{\text{image}} + \underbrace{\tfrac{1}{\sigma_2} \vec{u}_2^T \vec{w}}_{\text{noise}}) \vec{v}_2 + \ldots + \alpha_n (\underbrace{\vec{v}_n^T \vec{f}^o}_{\text{image}} + \underbrace{\tfrac{1}{\sigma_n} \vec{u}_n^T \vec{w}}_{\text{noise}})$$

(simple estimate)

α_i = either zero or one

want to keep terms where image is much larger than the noise

$\underline{\text{If}}$ $\left| \vec{v}_i^T \vec{f}^o \right| > \text{thresh} \left| \frac{1}{\sigma_i} \vec{u}_i^T \vec{w} \right| \Rightarrow \alpha_i = 1$ (thresh > 1)

$\underline{\text{Else}}$ $\alpha_i = 0$

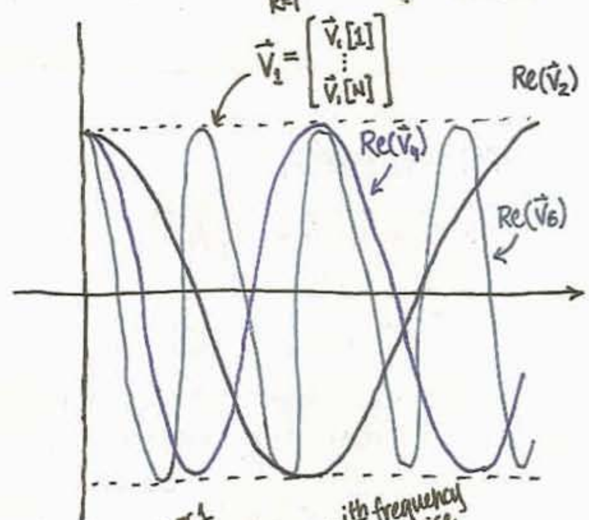
CONNECTION TO FREQUENCY RESPONSE

$$H_{CIRCULENT} = \overset{U}{\begin{bmatrix} \\ DFT^{-1} \\ \\ \end{bmatrix}} \begin{bmatrix} \times & & & \\ & \times & & \\ & & \ddots & \\ & & & \times \end{bmatrix} \overset{V^T}{\begin{bmatrix} \\ DFT \\ \\ \end{bmatrix}}$$

diagonal; complex numbers; DFT: discrete fourier transform

$$\frac{1}{N} \begin{bmatrix} | & | & \cdots & | \\ & e^{j2\pi\frac{(k-1)(n-1)}{N}} & & \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} X[1] \\ \vdots \\ X[N] \end{bmatrix} = \begin{bmatrix} x[1] \\ \vdots \\ x[N] \end{bmatrix}$$

$$x[n] = \frac{1}{N} \sum_{k=1}^{N} e^{j2\pi\frac{(k-1)(n-1)}{N}} X[k]$$



$$\vec{f}^{E} = \sum \alpha_i (\vec{v}_i \vec{f}^o + \tfrac{1}{\sigma_i} \vec{u}_i \vec{w}) \vec{v}_i$$

α_i: 0 or 1; $\vec{v}_i \vec{f}^o$: ith frequency in object; $\vec{u}_i \vec{w}$: ith frequency in noise

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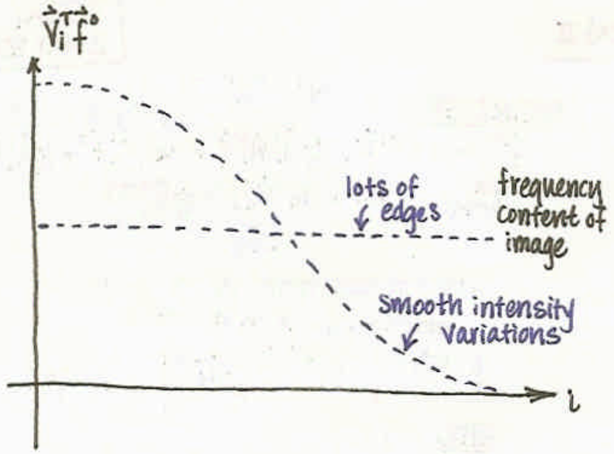
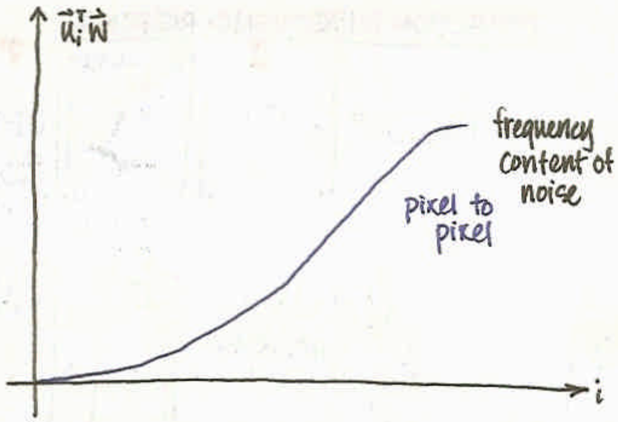
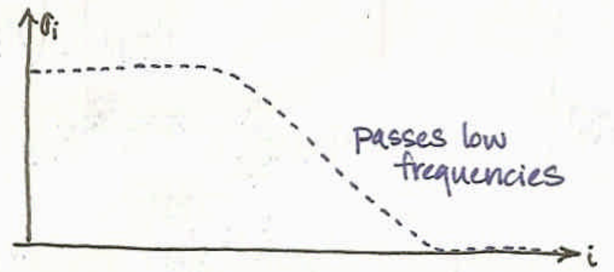
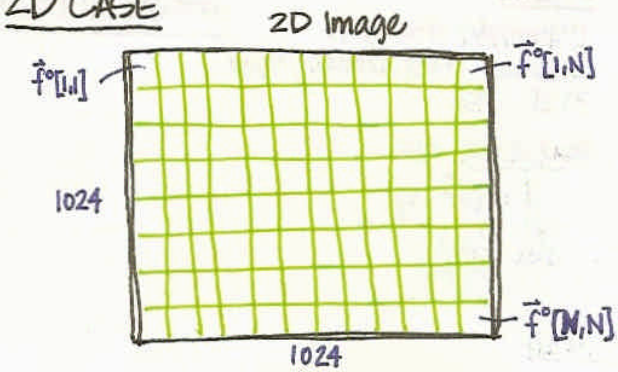
Foundations of Algorithms and Computational Techniques in Systems Biology
Professor Bruce Tidor
Professor Jacob K. White

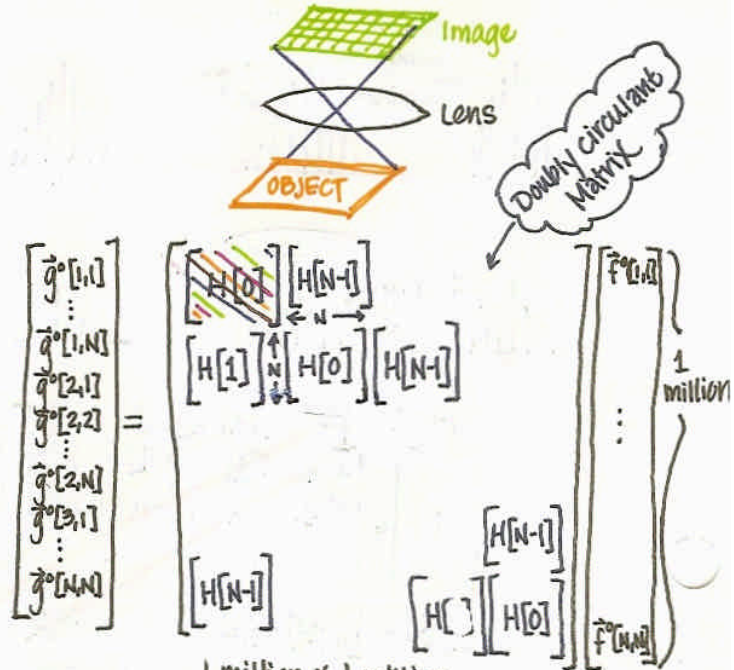






2D CASE



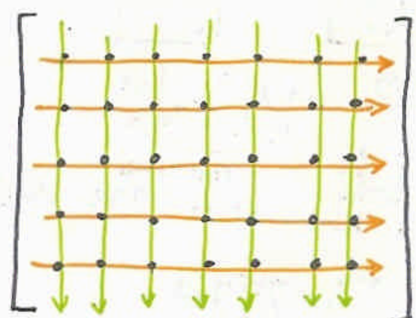


1 million x 1 million

(done fast with FFT)

2D DFT

$$X[n_1,n_2]=\frac{1}{N^2}\sum e^{-j2\pi\frac{(n_1-1)(k_1-1)}{N}}e^{-j2\pi\frac{(n_2-1)(k_2-1)}{N}}X[k_1,k_2$$



Which terms to keep

$$\vec{f}^E=\sum \alpha_i\vec{v}_i\left(\vec{v}_i^T\vec{f}^o+\frac{1}{\sigma_i}\vec{u}_i\vec{w}\right)$$

Is α_i 1 or 0?

$\vec{v}_i^T\vec{f}^o$ versus $\frac{1}{\sigma_i}\vec{u}_i^T\vec{w}$

Suppose we know the "energy" in the noise: $\vec{w}^T\vec{w}=E_{Noise}$
& suppose we also know the "energy" in the image: $\vec{f}^{oT}\vec{f}^o=E_{Image}$
$\Rightarrow$ energy in the image is uniform in frequency

$$\vec{f}^o=\sum\vec{v}_i(\vec{v}_i^T\vec{f}^o)$$

$$\vec{f}^{oT}\vec{f}^o=\left(\sum\vec{v}_i(\vec{v}_i^T\vec{f}^o)\right)^T\left(\sum\vec{v}_i(\vec{v}_i^T\vec{f}^o)\right)$$

$$\left|\vec{v}_i^T\vec{f}^o\right|=\frac{\sqrt{E_{Image}}}{N}$$

Uniform Noise assumption: $|\vec{u}_i^T\vec{w}|=\frac{\sqrt{E_{Noise}}}{N}$