QIQI WANG: Any questions so far? Especially if you have any additional questions on the ODE part, if anything is not clear I'd like to have them answered before we move on to PDEs. And you're going to see other things we do on PDE really depend on ODEs or prerequisite ODEs.

AUDIENCE: [INAUDIBLE].

QIQI WANG: Oh yeah. Good idea. Let me open it using my Chrome. Because the [INAUDIBLE] didn't work. OK. OK. These are the measurable outcomes. These also updated. So you have the ODEs. Now you scroll down. You have the Product Differentiated Equations. So we can look at all the measurable outcomes again. And for each of them, you can click on them. And it'll show you all the sections that you can use by yourself on how to navigate through the materials and make sure you have learned what we want you to learn.

All right. We have an identifier with the PDE using [INAUDIBLE] law. And that is that one of the things we are going to be talking about today. Qualitatively describe the solution of simple ODEs. And that is also something that we are going to be describing today. But you can read all of these also in the readings.

OK. Now let's dive into PDEs. OK. PDEs versus ODEs. When we solve for ODEs, we are looking at solution, we are discretizing a function of time. Right? So this is when we solve ODEs and when we want to aproximate-- The difference between an ODE and [INAUDIBLE] equation is because it compares a differential operator d/dt. And a lot of things we'll learn is how to approximate this d/dt and how we use the approximation to solve the ODE.

Now once we go to PDEs, there are two cases. One case is now I have a U function of both space and time. So we have space and time. That gives me a PDE. And now when we take derivatives, there are two derivatives we can take. We can take the positive derivative of U with respect to space. And the reason it is a positive derivative is because when you fix all the other independent variables-- so it is a derivative to x while holding time fixed. Or you can take partial derivatives to t. That is, taking derivatives to time while fixing the spatial location. This is very important because we are going to see when we also can take derivatives of this function of space and time, [INAUDIBLE] neither hold space nor time fixed. So we can also take something like a directional derivative when we are both moving in space and time. And that is something we are going to see in the characteristics. We can take the derivative in interaction and call it a characteristic reaction. And that is going to give us some interesting results.

Another possible case for PDE is when there are multiple spatial locations, we can have U of x and y or U of x, y, and z. And this is also a PDE because we can take the derivative to x. We can take-- and of course, in most cases you can have a function of x, y, and z. And t. That gives you a four-dimensional solution you can solve numerically. And a lot of the simulations, the aerodynamics simulations if you want to simulate the flow around an airplane, it's actually the unsteady flow air on an airplane, you do have to look at this case. You have three spatial dimensions and time. And you need to solve partial differential equations involving these.

And the mass per volume is what? It's density, right? The unit amount of momentum per volume is momentum density. The unit amount of energy per volume is energy density. So these are the quantities we solve. When these quantities can be spatially dependent-- the density of air, for example, in a compatible flow can be different from one spatial location to another. In an unsteady compressible flow, it can be different at different times. So this kind of view is going to be a function of space and time.

OK. So how do we express this kind of conservation law in terms of partial differential equations? First of all, what does it mean by, let's say, mass being conserved? So let's take a special case where U is the density. So let's say the unit amount of mass, the volume. So can somebody tell me what does it mean by mass being conserved in the mathematical sense?

AUDIENCE: [INAUDIBLE]

QIQI WANG: OK. So yeah. That's a very good answer. So first of all, you are choosing a controlled volume, right? So first of all, you are choosing a control volume. Let's say do my control volume, let's actually do the control volume like this. It can be an arbitrary control volume. Let's call it omega. So mass being conserved means the amount of mass inside the control volume is not going to magically appear or magically disappear. The increase has to be coming from outside. And the decrease has to go outside. Right? That means--

And also let me express what the mass is. The mass is the integral of the density integrated over the volume. Let's express it as this integral so that it makes more sense. Now this is the mass inside the control volume at a certain time. The time derivative of this mass which represents in a very small amount of time how much mass increased or how much mass has decreased inside the control volume. It has to be equal to the rate of mass coming into the volume minus the rate of mass going out of the volume. Right? And the mass going in and out can only happen on the surface of the control volume, on a boundary of the control volume.

And let me write like this. Plus the rate over the surface of control volume. Let me call it partial omega. This is a notation of the boundary of the control volume omega. It has to be equal to an integral at the boundary of the control volume times the flux of row. Let me call it f of row d S. That will be equal to zero.

OK. Let me expand this a little bit more. So this [INAUDIBLE]. It represents for a unit amount of time at a unit surface. How much mass has gone through that surface? It is directional because there is a direction of the flow of mass, right? For example, if you know it's a fluid flow, then in fluid flow f of row is going to be equal to row times the local velocity, right? So if you have a velocity vector and you know the density, then the unit among the mass going through a surface is equal to the row U times the area of the surface normal to the velocity. Right? So that basically is what we mean by a quantity being conserved. All Right.

Now this is not a differential equation, right? Anybody have a question? Question on this conservation law? OK. This is not a differential question yet. Right? And honestly I don't really know how to solve it. So how do we convert this into a differential equation that we can solve?

AUDIENCE: [INAUDIBLE].

QIQI WANG: Good point. We can shrink this omega to something very small. And this is exactly what we are going to see in the finite volume approach. So in the finite volume approach, we are actually going to shrink this omega to something very small and use this form to solve this equation.

Another method we're going to discuss is finite difference. So we're going to be teaching three methods. Finite difference, finite volume, finite element. So let's leave finite element to the last because it's the most advanced method. Finite difference. I'm trying to avoid, we're going to be discussing first. And finite volume is exactly taking that equation and shrinking omega to very small. The finite difference approach, on the other hand, has to first convert this equation into a differential equation. In finite difference, we do not allow any integrals. We have to express everything through derivatives. Now how to do that?

[INAUDIBLE] theorem or divergent theorem, right? With divergent theorem, we can convert the surface integral into a volume integral inside omega of the divergence of the flux. Right? So the conservation equation becomes like this, it becomes-- and we can also take the timed derivative inside of the fixed control volume. So we have integral over a really arbitrary domain, omega. Because this conservation law has to satisfy for any control volume. So this is the time derivative then. And we copy the divergence term into here. The x has to be zero. And this has to be zero for any control volume. For any omega.

That means what is inside the integral has to be zero. Because if the integral is non-zero anywhere, then we can make a very small control volume around it. And that integral is not going to be zero. So the integral has to be zero everywhere.

Now this is a differential equation, right? This is a differential equation. And it is a partial differential equation because we not only have derivatives in time, we also have divergence. And divergence is a derivative in space. So this is my PDE in the differential form. Right. This is my PDE in the differential form. And that is what we have over here, differential consideration law of U.

So a differential conservation law of U. In general, we have derived the conservation of mass, right? In general, the differential form of the conversion law is dU/dt plus the divergence of some flux which is a function of U is equal to a fourth term. And the fourth term is going to be non-zero for things like, for example, if you have energy-- if you have momentum-- for example, if you use momentum, momentum density, then the fourth term will be external force like gravity. Right? So you can also have a [INAUDIBLE]. So when we talk about a differential form of a conservation law, we are talking about equation [INAUDIBLE] like this.

All right. Let's take a look at a few examples. So the first example is what we do a lot of research on. The conservation of mass, momentum, and energy with flows. And by solving differential equations, that governs the [INAUDIBLE] conservation of three things-- mass, momentum, and energy-- we can really explain a lot of phenomenon that we are interested in in aerospace engineering. Like the flow around airplanes, rockets, and missiles. And also flow inside [INAUDIBLE] engines. We can all solve these problem by solving differential equations that just governs the conservation of mass, momentum, and energy.

All right. So this is one of the prime examples of conservation loss expressing partial differential equations. Another example of conservation law is wave propagation. So this is like surface waves. And many of the simulations you can see actually inside [INAUDIBLE] where we are solving partial differential equations that govern really the conservation of mass and momentum of water, that is under the force of gravity.

Another type of conservation law is what you find when you open a website and look at the weather forecast. So how do people predict the weather? They solve differential equations. They solve differential equations that govern the conservation of mass and momentum, weather molecules, energy. Of course, you have [INAUDIBLE] from the sun. And you have things from the ocean, things like that. But essentially it is a differential equation governing the conservation of stuff that happens on the Earth's atmosphere. Of course, it's a very large scale PDE. But you look at the weather forecast because people are able to solve PDEs.

So this is what we are going to be looking at in the PDE section of this class. We look at how to numerically solve partial differential equations. OK. So first of all, we are going to be looking at some examples of conservation laws. And these examples are the very elementary type of conservation loss that a lot of them, you should be able to qualitatively describe the solution, how the behaves. But really in complex differential equations like this, it's really the combination of multiple of these elementary conservation laws. And if you are able to solve each of these elementary conservation laws and you know how to combine them together and how to do that in complex geometry, we can really solve these very complex differential equations.

So the first conservation law is advection. Vachon And also many people in this field called convection. So convection is a physical phenomenon that is usually created by the bulk movement of molecules inside some fluid. Like, for example, the generation of the hurricane is really a convective phenomenon. So let's look at-- at the most elementary form-- what convection is like.

So let me go over here. I have some MATLAB [INAUDIBLE]. OK. And I want you to look at the first code there. But I'm going to give the source code to you via Dropbox. So here, what I'm doing is-- OK. I'm opening in MATLAB a window that has nothing in it. I like some of you to come up and design what's called an initial condition for this [INAUDIBLE]. All right? And let me first describe what the advection equation does. The advection equation describes the evolution of a concentration. Let's say mass density of a certain species. Density of, let's say, a pollutant or [INAUDIBLE]. Then how does the density evolve under the bulk movement of a fluid?

So convection, that is one field we are going to be going over here, describes the initial distribution of that species. Or the initial density of the species. And after [INAUDIBLE] is in a position, we are all going to be observing how does the distribution go under the bulk movement of some fluid? There's somebody here [INAUDIBLE]. Somebody is speaking at the back?

AUDIENCE: [INAUDIBLE].

QIQI WANG: Come on.

[LAUGHTER]

OK. Somebody--

[INTERPOSING VOICES]

QIQI WANG: [INAUDIBLE], people.

AUDIENCE: [INAUDIBLE].

QIQI WANG: OK. Thank you.

AUDIENCE: [INAUDIBLE].

QIQI WANG: Come up and draw it.

AUDIENCE: [INAUDIBLE].

QIQI WANG: You want to move [INAUDIBLE] starting from the left to the right.

AUDIENCE: OK.

QIQI WANG: And it's going to be-- try to make it periodic. [INAUDIBLE] on the screen.

AUDIENCE: [INAUDIBLE].

[LAUGHTER]

AUDIENCE: You want it to be periodic?

QIQI WANG: Yeah.

AUDIENCE: OK.

QIQI WANG: Going from the left to right, we want [INAUDIBLE]. Yeah.

AUDIENCE: OK.

[INTERPOSING VOICES]

QIQI WANG: Let me try it again. Let me try it again.

AUDIENCE: Is it going to freak out if it's negative?

QIQI WANG: No, no. It's not going to freak out. This [INAUDIBLE] with most [INAUDIBLE].

AUDIENCE: OK.

QIQI WANG: Good.

AUDIENCE: That wasn't--

QIQI WANG: Keep going. Keep going. Keep going. Keep going.

AUDIENCE: It's not periodic.

QIQI WANG: It's fine, it's fine.

AUDIENCE: OK.

QIQI WANG: All right. OK. We actually have two windows. That's why it's-- so let's look at the right window first. Thank you. OK. Let's look at this window first. So this is the evolution of the equation after a certain time, right? It's accelerating, right? This is the solution to the partial differential equation at a certain point. So this is the spatial coordinate x. And the time is not typical. This is the [INAUDIBLE]. And this [INAUDIBLE] over here. [INAUDIBLE] is how the solution evolved. What is this solution doing? Yeah?

AUDIENCE: It's moving all the [INAUDIBLE].

QIQI WANG: OK. The answer is-- the solution, kind of, moves while also [INAUDIBLE]. But here the template [INAUDIBLE].

AUDIENCE: [INAUDIBLE]

QIQI WANG: Yes. Good observation. The amplitude has to be the same. Right? 0.6 minus [INAUDIBLE]. If x equals stays the same and the whole thing just moves towards the right at a fixed velocity. So this is a solution to the pure advection equation. And it is the solution that keeps moving towards the right without diffusing. So the question is partial U-- let me call it small u-- partial small u, partial t plus a big U partial U-partial x equal to zero. Here there is a big distinction between the small u which is a function of space and time, is the unknown which is also what we plotted. And the big U is neither function of space nor time. It's a constant. OK? So big U is a constant advective velocity. And small u is the unknown.

So two things. One is, is this equation a conservation law? Do you think, just from guessing from the solution you saw, is that a conservation law? Yes. Because it looks like the density or the solution is conserved, right? It just moves around. Whenever that goes into-- if you look at a control volume, that's because [INAUDIBLE]. Whatever that [INAUDIBLE] right? So it is a conservation law. Mathematically, how does that come into a conversation law? Mathematically if I write the equation a little bit differently, if I write the equation differently-- partial u-partial t plus partial F of u partial x is equal to zero where the F of u, the flux of u is just equal to big U times small u. Right? Does that look more like a conversation law?

Why? Because we already mentioned that derivative in x is-- Yeah. This is what the divergence is having a multiple dimension, right? The divergence of [INAUDIBLE] is e x component of the method dx plus the y component of the method dy. And in one dimension there is no dy. So it would be dx is the divergence. That's what happens in one dimension. The divergence is [INAUDIBLE].

And now the use of flux. The flux is equal to big U times small u. That makes sense, right? Because if the small u is the solution [INAUDIBLE], then the [INAUDIBLE] of the density should not be the velocity times the local density. Right? The higher the velocity is, the faster the mass goes through [INAUDIBLE]. The higher the density is, also the faster the mass goes through [INAUDIBLE]. The flux indicated is velocity times the [INAUDIBLE]. Right?

OK. Now there is a conservation law, right? It is a conservation law. And it seems it has an analytic solution of everything that's moving towards the right. Is that true? Yes. We do have an analytic solution. And the analytic solution looks like this. Let's take a look at this again. So this is the solution at a particular time-- in this case, it is at time 0.5. But if you don't know the solution at a certain time, the entire solution as a function of space and time, it looks like this. It looks like this.

Now over here I already have this-- and [INAUDIBLE]. The solution as [INAUDIBLE]. But other solutions [INAUDIBLE]. OK. Blue is negative value and red is positive value.

[INTERPOSING VOICES]

QIQI WANG: Here you can see that at time equal to zero, this is [INAUDIBLE]. This is the initial condition. As time increases, let's say, at 0.1, the peak has gone from here instead of here at the initial condition. At the initial condition, the peak is here. The bottom is here. [INAUDIBLE] the peak has moved towards the right. The bottom has moved towards the right. Everything has moved towards the right. And it goes over the [INAUDIBLE]. And we're going to discuss later, it depends on the conditions [INAUDIBLE]. And keep moving towards the right. All right.

And you'll see the lines over here, the lines which really is the contour of the solution for space and time-- these lines are called the characteristic lines in the solution of a partial differential equation. OK. So why are they important? Why does it have a name? Do we see something particular about these lines?

AUDIENCE: [INAUDIBLE].

QIQI WANG: Huh? They all have the same flow or-- first of all, they are straight, right? We all see the contour of the random function of space and time, you would expect the [INAUDIBLE] to be a curve. Right? But these contours are straight. And we're going to see later on that [INAUDIBLE] another equation we happen to be looking at. The second is, the solution is [INAUDIBLE] along these lines. And these are really the characteristics of characteristic lines. And in this case, we can analytically derive what they are because there is an analytical solution of this differential equation. U of x and t is just equal to whatever initial condition is-- let me call u naught the initial condition-- x minus U times t.

OK. What does this mean? It means-- remember u zero of x minus Ut. It means at t little zero, it makes sense that at t little zero, my solution U of x, t-- that is [INAUDIBLE]. That is my initial condition. When t increases, the solution at a particular x is the initial condition at a smaller x. Let's assume U is positive. If U is positive, then at a positive t, my solution at a certain x is equal to the initial condition at a smaller x. That make sense? At a particular time, my solution at one x is equal to the initial condition at a smaller x. And how much smaller it is really equals U times t [INAUDIBLE] is [INAUDIBLE] of how fast these lines go towards the right. And how fast those lines go towards the right is determined by U. It's determined by how fast the waves are convecting towards the right.

Any questions on this equation? So this is good example of a partial difference equation. Because we know its analytic solution. And when we use a numerical MATLAB to solve the differential equation, we can compare against this analytic solution. It is like when we look at all these, we want to try our numerical methods on du/dt for the lambda U. Not because it is the most useful equation to solve-- I mean, it is useful but the reason we want numerical methods is because we want to apply this to more complex equations-- the reason we want numerical methods for PDEs is because we want to solve free-flow equations. But in order to have our methods and see how our methods behave, to analyze what method is useful to apply our methods, we're not going to have equations like this. This one of the elementary types of differential equations-- the convection equation or advection equation.

All right. And so the characteristic lines are x equals to some x zero plus U t. So this is the characteristic line. And we are going to also see similar characteristic lines later on. OK.

Let me give you another example of conservation law. Now this time, the equation we are solving is no longer linear. It's called the Burgers equation. It is the equation people use to model fluid flows in the higher fidelity than the advection, than the linear advection equation. And one of the key features we're going to see in this equation is shock waves. Shock waves as we see in supersonic [INAUDIBLE] that develop shock waves.

OK. So the bulk of the equation looks like this. Partial t plus u partial u-partial x equal to 0. It looks the same as what we did before. But there is a one key difference in what we did before. This U is a constant that has nothing to do with this solution. In both of the equations, this U was not really a solution. So the bulk of the equation had a quadratic [INAUDIBLE] because this one is a [INAUDIBLE] which will make U twice as large. That [INAUDIBLE] is going to be four times as large. All right?

OK. Is this equation a conservation law at all? Yes. Can we write the equation into a form that looks like this? Remember in order to write in conservation form, we have to have an equation like this. Right? Can we write this equation in some form with somehow defined F? One half of U squared exactly. OK. Because if you take this and wrap this into the [INAUDIBLE] the derivative of U squared, it is going to be two times U times the u. And the [INAUDIBLE] is [INAUDIBLE] here. You get back to the first equation.

OK. Now let's see how this equation behaves differently from the linear advection equation.

Can I get somebody else to draw me another solution? Thank you.

AUDIENCE: [INAUDIBLE].

QIQI WANG: Start on the left to right to the [INAUDIBLE] All right. Thank you. So let's see how this solution goes. And [INAUDIBLE].

AUDIENCE: [INAUDIBLE].

QIQI WANG: Sorry?

AUDIENCE: [INAUDIBLE].

QIQI WANG: What do we see that is different from what we saw in the linear advection equation?

AUDIENCE: [INAUDIBLE].

[INTERPOSING VOICES]

QIQI WANG: Huh?

AUDIENCE: The magnitude.

QIQI WANG: The magnitude is [INAUDIBLE]. But [INAUDIBLE].

AUDIENCE: [INAUDIBLE].

QIQI WANG: The bottom is coming up a little bit. And the top is going down a little bit. Right? So there is something that [INAUDIBLE]. Right? This is one of the things we didn't see in the previous equation. We have developed a [INAUDIBLE] right? And [INAUDIBLE] What else do we see? Does the wave go towards the right at a uniform speed?

[INTERPOSING VOICES]

QIQI WANG: [INAUDIBLE]. Normally we have something that moves toward the right, we [INAUDIBLE] the solution has been [INAUDIBLE]. OK. And if we go back to this one, looking at the solution of the function of space and time, we can see that we no longer have parallel characteristic lines. But we still have these characteristic lines. Right? And there you will see other things [INAUDIBLE]. This is initially not a shock wave, but now it has become a shock wave. A shock wave is not [INAUDIBLE]. All the characteristic lines that are not shock waves [INAUDIBLE].

[INAUDIBLE] characteristics are [INAUDIBLE]. All right.

So OK. Let's let the animation go. And let's take a five minute break. And after the break, we're going to be looking at a more, we're going to be sort of animating these solutions all ourselves. And we'll see what we get in terms of the solution.

OK. So let's-- we have seen some solutions of partial differential equations. All right. So right now what I'd like you to do is have everybody come to the front of the classroom. And we are actually going to be emulating the solution of the differential equation.

AUDIENCE: [INAUDIBLE].

[LAUGHTER]

[INTERPOSING VOICES]

[INTERPOSING VOICES]

QIQI WANG: OK. Let's stand so that the two doors are boundaries to this. So let's [INAUDIBLE].

AUDIENCE: [INAUDIBLE].

QIQI WANG: Yes. Sure. Can somebody there just push the off button? All right. OK. So let's [INAUDIBLE] ourselves a little bit? [INAUDIBLE] so that we can go in and out of the door.

[INTERPOSING VOICES]

QIQI WANG: Let me just have--

AUDIENCE: [INAUDIBLE].

QIQI WANG: Just in case.

AUDIENCE: We're worried about [INAUDIBLE].

AUDIENCE: Yeah.

[LAUGHTER]

QIQI WANG: Good point.

[INTERPOSING VOICES]

QIQI WANG: OK. So let's [INAUDIBLE]. So let me get one student here that's going to be an observer. OK.

[LAUGHTER]

OK. So if you stand over here, you can really [INAUDIBLE]. Let's say the height [INAUDIBLE] is the solution of my PDE. OK.

[LAUGHTER]

[INTERPOSING VOICES]

QIQI WANG: OK. So I have a distribution like this. This is my solution. [INAUDIBLE] space. Right? And imagine I have this [INAUDIBLE]. So what I want to do is I want to simulate the solution of an advective differential equation. So when I say I'm going to the next concept, let's think of-- we won't move all at one time. Right? Let's just move our position. What do you think we should do when we evolve to the next time slot?

[INTERPOSING VOICES]

QIQI WANG: Move that way. Somebody has to be emulate the periodic--

[LAUGHTER]

When I go, let's go. Go. One step. OK. Step one step. Oh it doesn't [INAUDIBLE]. OK. Two steps. OK. So let's go there for two more steps. All right.

[LAUGHTER]

OK. Now it's a perfectly good solution because everybody moves there at a constant speed, right? Let's stop for a moment. Let's think of emulating the solution of a Burgers equation.

[INTERPOSING VOICES]

AUDIENCE: Does anybody feel like they're on a [INAUDIBLE]?

[LAUGHTER]

QIQI WANG: So we haven't talked about diffusion yet. We're going to be talking about that later. But in

Burgers equation, what happens is that when we look at the characteristic lines-- [INAUDIBLE] actually following the characteristic lines. Because your height is going to stay constant, right? As long as [INAUDIBLE] your height is going to stay constant. Now what is different is that in the purely advective equation, the characteristic lines stay parallel. That means all of you-- no matter of your height or no matter what solution value you are at-- are going to be moving at the same speed. Right?

In a Burgers equation, what is difference? The solution-- remember we have U times du/dt. And the U in front of the du/dt is no longer constant. It is actually the solution itself. So what does it mean? How fast should you move when you are emulating the [INAUDIBLE] equation?

AUDIENCE: Four [INAUDIBLE]?

QIQI WANG: Exactly. The [INAUDIBLE] are going to be moving. It's going to be proportional to the solution itself. So in other words, if you're [INAUDIBLE] for the solution, then the speed you move is going to be actually proportional to how tall you are.

[LAUGHTER]

OK. So let me do this. When I say-- when I say step, let's just move for a single step. And let's say the speed, the step size we're going to take, let's say, is-- let's estimate-- is like a third of your height. Let's try to see if you can--

AUDIENCE: OK.

--estimate how to do that. And whoever goes out of the door has to come back as a periodic [INAUDIBLE].

[INTERPOSING VOICES]

QIQI WANG: OK. Everybody ready? One step.

[INTERPOSING VOICES]

QIQI WANG: OK. [INAUDIBLE].

[LAUGHTER]

OK. Another step.

[LAUGHTER]

OK.

AUDIENCE: Um?

QIQI WANG: Yeah?

AUDIENCE: Can characteristic lines cross each other? Or when they're right in front of each other?

QIQI WANG: OK. We have a good question here. Can characteristic lines cross each other? Because look at here. What happened here?

[LAUGHTER]

So what is the physical phenomenon that is happening at this point?

[INTERPOSING VOICES]

QIQI WANG: OK. Well what physical phenomenon is happening here?

AUDIENCE: We have a shock wave.

[LAUGHTER]

QIQI WANG: We have a shock wave. Right? We have a shock wave. Because its characteristics in front is moving slower than the characteristics at the back. So instead the molecules at the back catch up. And you get a shock wave forming. So actually once you have a shock wave forming, [INAUDIBLE] you all should move at the same time. You also move at the same speed that is equal to the average of your speed.

OK. So another step.

[INTERPOSING VOICES]

QIQI WANG: Oh there is another shock wave forming over here. OK. Another step.

[INTERPOSING VOICES]

QIQI WANG: OK. Another step. Another step.

[INTERPOSING VOICES]

QIQI WANG: All right. So we have a lot of shock waves, right? A lot of shock waves. And we actually get either shock waves or a solution that is very smooth, right? So we end up getting a solution not unlike what we saw previously on the screen. So when the characteristic in front moves back up, they spread out. And the result of the solution becomes smooth. That is like an anti-shock wave. It's a verification wave. It helps when you have supersonic expansion. [INAUDIBLE] smoother.

On the other hand, if the solution at the back is faster, then it catches up. And the characteristic lines converse each other. And [INAUDIBLE] form these continuities which is shock waves.

OK. OK. Do we want to move more? Or--

[LAUGHTER]

Or do we understand how the differential equations behave?

[INTERPOSING VOICES]

QIQI WANG: OK. Propagate. Let's go--

[INTERPOSING VOICES]

QIQI WANG: Traffic jam. OK. That's fine. That's good enough. Let's go back to our seats.

[INTERPOSING VOICES]

QIQI WANG: And interestingly the bulk of the equation looks something really similar to the Burgers equation. If we use the model of the flow of cars on highways, or freeways, so imagine the lots of cars-- [INAUDIBLE]. Cars are conserved, right? Think of a highway that has no entrance and exit. The The cars are conserved. No car disappears, no car is generated like spontaneously. So the number of cars is conserved. And the flux, which is proportional to the density of the cars and also proportional to the speed of the cars-- and the speed of the cars can usually depend on the density, right? If you have really a lot of traffic, the cars move slowly. If you have not a lot of traffic, they will move at slightly faster than the speed limit. So really the flux depends on density.

So you really have conservation laws. And you have behavior similar to this. But in the case of traffic flow, the shock waves actually move backwards. So think you have a red light, all the cars kind of bumping into each other and stopping. We have a shock wave propagating backwards. So this kind of equation is not exactly the Burgers equation but something pretty similar can model the evolution of the density of cars on a highway.

And maybe we can take a look at the Burgers equation again. And who was the observer? OK. You observed. Can you tell us what [INAUDIBLE] what the initial distribution of the height was?

[LAUGHTER]

And we can kind of a look at the solution evolution again, just try to recall where you were. And was the solution kind of reproducing the kind of shock wave [INAUDIBLE] a little bit as well?

AUDIENCE: OK. There was a group of tall people over here. And some shorter ones. A few tall ones.

[LAUGHTER]

A few shorter ones.

QIQI WANG: All right.

[CHATTER]

QIQI WANG: Yeah. I see kind of a-- these [INAUDIBLE] performed shock waves really quickly. Oh, these I don't have [INAUDIBLE]. Sorry. So there are those people coming to the door. I think those are [INAUDIBLE].

[LAUGHTER]

So yeah. [INAUDIBLE]. There is the replication wave over here. And you can see the [INAUDIBLE] shock waves already formed over here. So there are already people [INAUDIBLE]. And here because there is a continuous [INAUDIBLE] over here, you can already see the [INAUDIBLE] are converging. Initially they just spread out. Now they are closer with each other. And [INAUDIBLE] that have already formed.

OK. Let's wait until this shock wave forms [INAUDIBLE].

AUDIENCE: I think that's [INAUDIBLE].

QIQI WANG: OK. So let's see the response and we can move onto the diffusion equations. Yeah. The diffusion equation, I think, is much more difficult to act. Because we can't really diffuse our height into people around you.

[LAUGHTER]

So if you think of a way to do that, I'll do it next year.

[LAUGHTER]

OK. Now the shock waves are converging each other. And now these three guys are moving at the same speed as each other. OK. All right. And sorry for not having periodic on the conditions.

So now let's look at the Burgers equation. The Burgers equation has characteristics that are x equal to some x zero plus my U times t. And U is the local solution that stays the same along the characteristic. OK. Why is it like this? Because if you look at the derivative of the solution along the characteristic line, so du/dt-- let's look at this. Let's look at the solution that we have like this, right? Let's look at the derivative of the solution along this characteristic line.

So we're looking at du/dt. Notice I'm using the total derivative because I'm looking at the solution along the characteristic which is at spatial location x zero plus Ut and at time t. So I'm allowing my space and time to change together as t increases. I'm not fixing the space and taking the derivative through time. I'm taking the derivative through time while the space depends on time as I go along this curve. So this derivative is like taking the derivative of yourself as you move in, right? And it's like taking the time derivative of a moving point in the solution.

So now what is it? Anybody remember the chain rule? How do you apply the chain rule to expand the total derivative into the partial derivative? Sorry? x naught is a constant, yes. And looking at one particular characteristic. Hm?

AUDIENCE: [INAUDIBLE].

QIQI WANG: Uh huh. Du/dx? dx dt. And here my is x is this. So dx dt is U, right? So this is equal to U. And plus partial u-partial t. Because this is also a function of time. OK. Now if you look at this, du dt, du dt. du dx times U. U times du dx. So this is exactly equal to the left hand side of the partial differential equation. And according to the partial differential equation, it is equal to zero, right?

What does this mean? That means the derivative of this solution as we go along the characteristic is equal to zero. Or the solution doesn't [INAUDIBLE] along the characteristic. That is exactly what we saw in the behavior of the solution and exactly what we tried to emulate as a group along the classroom. And the behavior is derived from the analytic form of the differential question. And you can use the same math here to derive it in the advection equation. You can use the same math to derive similar things for all non-linear conservation laws, all scalar non-linear conservation laws.

You're going to get-- all of the characteristics are going to be different. The U is-- and you need to convert the conservation law into something like this. You have to say-- you have to take this out, which is really partial df du, you have to take this out. But [INAUDIBLE] as long as you have this df du, df du is the speed of your characteristic line. And you can derive the same fact that the solution doesn't change along the characteristics.

OK. So I only have a few minutes. Let me do another example. That is, what if we have advection and diffusion at the same time? So when I have advection and diffusion at the same time, then I need to specify how much convection I have and how much diffusion I have.

So let me start to build a case where I only have-- so let me say, first of all, I only have a case of-- so u is the coefficient on the convection. This is the coefficient on the-- kappa is the coefficient on the diffusion. Let's take a large-- just one convection and one diffusion. OK. Can somebody help me draw initial conditions? Come on. We just get less than five minutes left. All right. Thanks.

AUDIENCE: Are the [INAUDIBLE] periodic or does it matter?

QIQI WANG: It doesn't matter. All right. Ooh. OK. Whether you draw a pretty [INAUDIBLE], then some very [INAUDIBLE]. That is the color of the diffusion. OK. So you see that kappa equal to one is very strong diffusion. OK. Thank you.

OK. Let me try another one that is-- let me make kappa equal to 0.01. Can somebody come down to draw the last initial condition before we go home? Thank you.

All right. So now because we have a much smaller kappa tendency, the solution is smoothed out. Right? The initial discontinuity is down. And the solution becomes smoother and smoother and smoother. At the same time, its [INAUDIBLE] was the right periodically. But because the coefficient is small, the [INAUDIBLE] is pretty apparent. But the rate is much smaller. And the main impact is like [INAUDIBLE] and there is a smaller diffusion, but still it is visible.

All right. So this is the behavior of the equation with [INAUDIBLE] flux. All right? Any questions? No? All right. Then I will see you next Monday then. Or I'll see you on Friday. I'm going to announce the office hours. Yeah?

[INTERPOSING VOICES]

AUDIENCE: [INAUDIBLE].

QIQI WANG: Yeah. The project now only has two questions that is relating to ODEs. And I'm going to add another question that is going to be related to PDEs. All right.