16.90 Lecture: Method of Weighted Residuals

April 7, 2014

Today's topics:

0: MOs for this module.

1. Model problem: steady 1D diffusion

2. Solution approximation

3. Collocation method.

4. Method of weighted residuals

1. Model problem: steady 1D heat diffusion in a rod

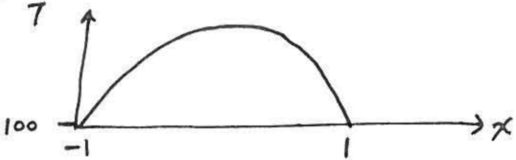
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -q$$

$k(x)$: thermal conductivity of material

$q(x)$: heat source / unit length

For $k=1$, $L=2$, $q(x)=50e^{x}$
$T(1)=100$, $T(-1)=100$

} analytical solution
$T(x) = -50e^{x} + 50x\sinh 1 + 100 + 50\cosh 1$



2. Solution approximation

(2a) Use a sum of weighted functions:

$$\tilde{T}(x) = 100 + \sum_{i=1}^{N} a_i \, \phi_i(x)$$

approx. of $T(x)$

chosen to satisfy BCs here

unknown weights

known functions ("basis functions")

Determine $\tilde{T}(x)$ (infinite dimensional) $\Rightarrow$ Determine $a_1, a_2, \dots, a_N$ (N unknowns)

Many choices for $\phi_i(x)$. Here we use polynomials (used in FEM)

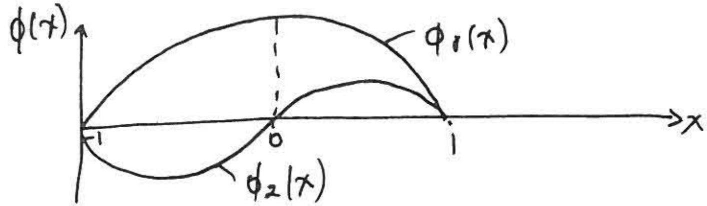
(2b) Choosing $\phi_i(x)$

Here need $\phi_i(1) = 0$, $\phi_i(-1) = 0$, $i = 1, .., N$

Linear function: get $\phi(x) = 0$ → trivial solution

Quadratic function: choose $\phi_1(x) = (1+x)(1-x)$

Cubic function: choose $\phi_2(x) = x(1+x)(1-x)$



With $N = 2$, our approximation is

$$\tilde{T}(x) = 100 + a_1 \phi_1(x) + a_2 \phi_2(x)$$

Now: how to determine a_1, a_2?

3. Collocation Method.

For $\tilde{T}(x) = 100 + \sum_{i=1}^{N} a_i \phi_i(x)$, need to determine $a_1, a_2, \ldots, a_N$

collocation: enforce PDE at N points.

1D heat equation: $\frac{d}{dx}\left(k \frac{dT}{dx}\right) = -q$

Define <u>residual</u> $R(\tilde{T}, x) = \frac{d}{dx}\left(k \frac{d\tilde{T}}{dx}\right) + q$

approx. sol^n

→ tells us by how much the approximate solution $\tilde{T}(x)$ does not satisfy the governing equation

→ not the same as error

→ note $R(T, x) = 0$

exact solution

Our example: $k=1$, $q=50e^x$

$$\tilde{T}(x) = 100 + a_1 \phi_1(x) + a_2 \phi_2(x)$$

$$= 100 + a_1 (1+x)(1-x) + a_2 x (1+x)(1-x)$$

$$\frac{d\tilde{T}}{dx} = -2a_1 x + a_2(1-3x^2)$$

$$\frac{d^2\tilde{T}}{dx^2} = -2a_1 - 6a_2 x$$

$$\Rightarrow R(\tilde{T},x) = \frac{d}{dx}\left(k\frac{d\tilde{T}}{dx}\right) + q = -2a_1 - 6a_2 x + 50e^x$$

residual

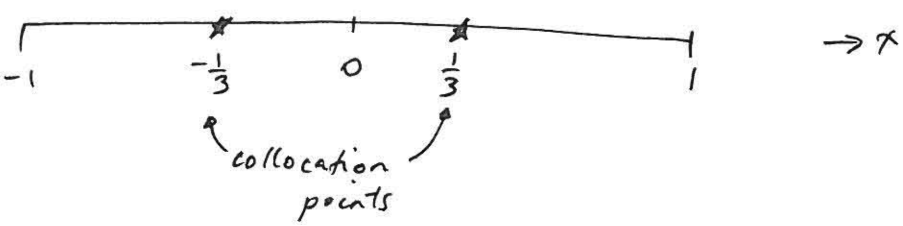
→ see that residual cannot be zero for all x

i.e., $\tilde{T}$ has some error

Need to choose values for a_1 and a_2

Collocation method: enforce PDE at $N=2$ points

i.e., set $R(\tilde{T},x)=0$ at two points x



set
$$\left.\begin{aligned} R(\tilde{T}, -\tfrac{1}{3}) &= -2a_1 + \frac{6a_2}{3} + 50e^{-1/3} = 0 \\ R(\tilde{T}, \tfrac{1}{3}) &= -2a_1 - \frac{6a_2}{3} + 50e^{1/3} = 0 \end{aligned}\right\} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 26.402 \\ 8.489 \end{bmatrix}$$

Our approximate solution is

$$\tilde{T}(x) = 100 + 26.4(1-x)(1+x) + 8.5x(1-x)(1+x)$$

⇒ Plots of residual, error vs. x.

4. Method of Weighted Residuals

Define a <u>weighted residual</u>

$$\underset{\substack{\uparrow \\ \text{ith weighted residual}}}{R_i(\tilde{T})} = \int_{-1}^{1} \underset{\substack{\uparrow \\ \text{weighting function}}}{w_i(x)} \, \underset{\substack{\uparrow \\ \text{residual}}}{R(\tilde{T}, x)} \, dx$$

MWR: Require N weighted residuals to be zero

→ choose N weighting fns $w_1(x), w_2(x), \ldots, w_N(x)$

→ get N equations to determine N unknowns $a_1, a_2, \cdots, a_N$

<u>Galerkin</u> method: choose $w_j(x) = \phi_j(x)$

(same functions used to approximate the solution and to weight the residuals)

For our example:

$$w_1(x) = (1-x)(1+x)$$

$$w_2(x) = x(1-x)(1+x)$$

$$R_1(\tilde{T}) = \int_{-1}^{1} w_1(x)\, R(\tilde{T}, x)\, dx$$

$$= \int_{-1}^{1} (1-x)(1+x)(-2a_1 - 6a_2 x + 50e^{x})\, dx$$

$$= -\frac{8}{3} a_1 + 200\, e^{-1}$$

$$R_2(\tilde{T}) = \int_{-1}^{1} w_2(x)\, R(\tilde{T}, x)\, dx$$

$$= \int_{-1}^{1} x(1-x)(1+x)(-2a_1 - 6a_2 x + 50e^{x})\, dx$$

$$= -\frac{8}{5} a_2 + 100\, e^{1} - 700\, e^{-1}$$

Set $\left.\begin{array}{l} R_1(\tilde{T}) = 0 \\ R_2(\tilde{T}) = 0 \end{array}\right\}$ $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 27.591 \\ 8.945 \end{bmatrix}$

$\Rightarrow$ Plots of residual, error