

Solution to 16.881 Homework#2

Quality Loss Functions

Dan Frey, 6 JUN 98

ORIGIN := 1 Let 1 be the first index in any vector.

1. Prove that the expected value of the nominal-the-best quality loss function is proportional to the second moment of the quality characteristic about the target value.

Solution:

The nominal-the-best quality loss function is defined as

$$L(y) = k \cdot (y - m)^2 \quad (1)$$

Expectation is defined as

$$E(f(x)) = \int f(x) \cdot p(x) dx \quad (2)$$

Substituting (1) into (2)

$$E(L(y)) = E[k \cdot (y - m)^2] = \int k \cdot (y - m)^2 \cdot p(y) dy \quad (3)$$

From the session 1 notes, the nth moment about the target (m) is

$$E[(x - m)^n] = \int (x - m)^n \cdot p(x) dx \quad \text{QED}$$

2. (60%) A plant manufactures 100 Ohm resistors. The acceptable tolerances are +/- 5% of the nominal value. The cost to scrap the product is \$0.10.

a) Assume a normally distributed variation...

Solution:

ORIGIN := 1 Let 1 be the first index in any vector.

$A_o := 10$ Cost to scrap the resistor [cents]

$m := 100$ Nominal value of the resistance (in ohms)

$\Delta_o := 5\% \cdot m$ Allowable variation in the diameter is +/- 5%

$$k_1 := \frac{A_o}{\Delta_o^2} \quad k_2 := \frac{A_o}{\Delta_o^2}$$

$$L(y) := \begin{cases} k_1 \cdot (y - m)^2 & \text{if } y > m \\ k_2 \cdot (y - m)^2 & \text{if } y \leq m \end{cases} \quad \text{Define the quadratic loss function}$$

$$y := m - 1.2 \cdot \Delta_o, m - 1.2 \cdot \Delta_o + \frac{\Delta_o}{10} .. m + 1.2 \cdot \Delta_o \quad \text{Define a range over y for the purpose of plotting}$$

Create a Monte Carlo simulation of the manufacture of the resistors.

$n := 1000$ Number of resistors to be manufactured

$$\sigma := \frac{\Delta_o}{3}$$

$$\mu := m + 0 \cdot \sigma$$

$R := \text{rnorm}(n, \mu, \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

number_of_bins := 20

width_of_bins := $\frac{4 \cdot \Delta_0}{\text{number_of_bins}}$

j := 1.. number_of_bins + 1

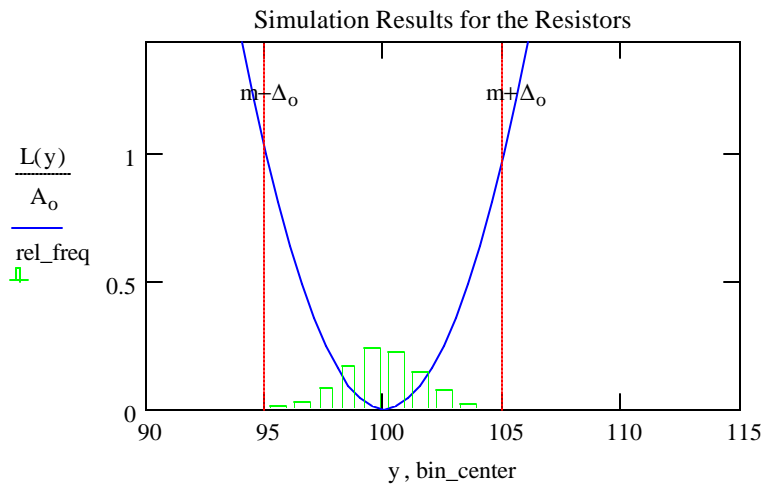
bin_j := $m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$

Define a vector with the start and end points of the bins.

rel_freq := $\frac{\text{hist}(\text{bin}, R)}{n}$

Compute the relative frequency distribution over interval.

bin_center := bin + 0.5 · width_of_bins



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 1.081 in cents

How does this compare to the theoretically derived figure?

$$\int_{m-2 \cdot \Delta_0}^{m+2 \cdot \Delta_0} k_1 \cdot (y - m)^2 \cdot \left[\frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-(y-\mu)^2}{2 \cdot \sigma^2}} \right] dy = 1.111 \quad \text{in cents}$$

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 1.111 \quad \text{Or, according to Phadke 2.10.}$$

b) Change to "Six Sigma"

Create a Monte Carlo simulation of the manufacture of the resistors.

$n := 1000$ Number of resistors to be manufactured

$$\sigma := \frac{\Delta_0}{6}$$

$$\mu := m + 0 \cdot \sigma$$

$R := \text{rnorm}(n, \mu, \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

$\text{number_of_bins} := 20$

$$\text{width_of_bins} := \frac{4 \cdot \Delta_0}{\text{number_of_bins}}$$

$j := 1.. \text{number_of_bins} + 1$

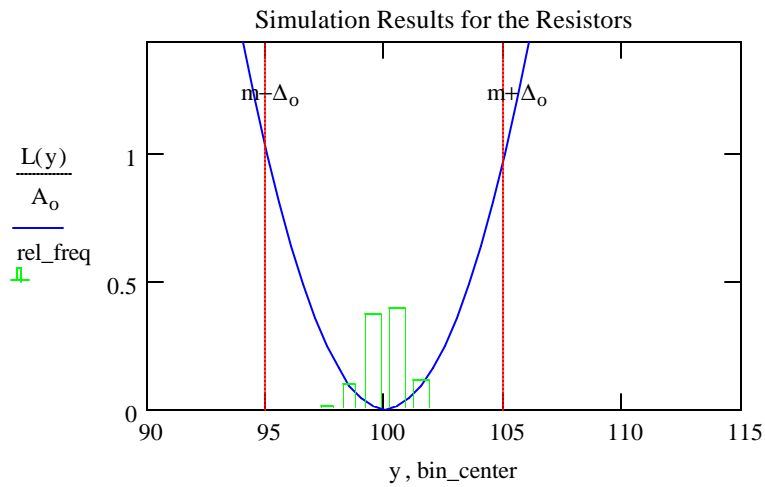
$$\text{bin}_j := m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$$

Define a vector with the start and end points of the bins.

$$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n}$$

Compute the relative frequency distribution over interval.

$$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 0.266 in cents

How does this compare to the theoretically derived figure?

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 0.278 \quad \text{Or, according to Phadke 2.10.}$$

A drop from about 1 cent per 10 cent resistor to 0.28 cents per 10 cent resistor. This is fairly dramatic, but not as dramatic as the changes in defect rate.

c) Consider the effect of a 1.5 sigma mean shift

First consider the effect on the six sigma process.

Create a Monte Carlo simulation of the manufacture of the resistors.

$n := 1000$ Number of resistors to be manufactured

$$\sigma := \frac{\Delta_0}{6}$$

$$\mu := m + 1.5 \cdot \sigma$$

$R := \text{rnorm}(n, \mu, \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

$\text{number_of_bins} := 20$

$$\text{width_of_bins} := \frac{4 \cdot \Delta_0}{\text{number_of_bins}}$$

$j := 1.. \text{number_of_bins} + 1$

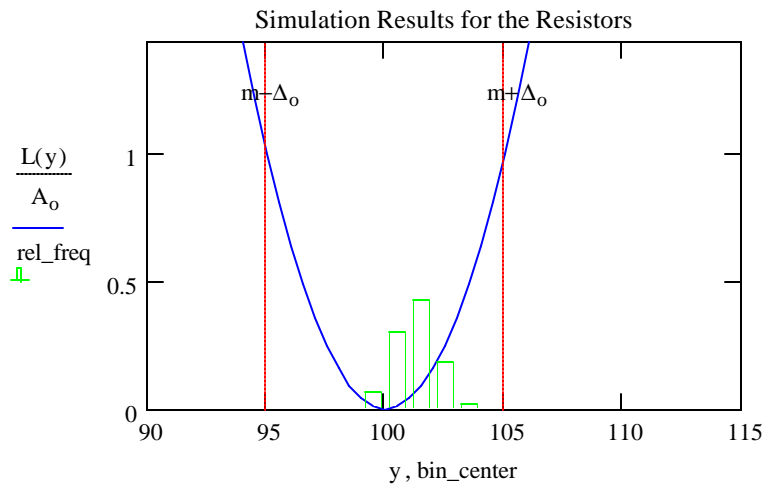
$$\text{bin}_j := m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$$

Define a vector with the start and end points of the bins.

$$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n}$$

Compute the relative frequency distribution over interval.

$$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 0.95 in cents

How does this compare to the theoretically derived figure?

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 0.903 \quad \text{Or, according to Phadke 2.10.}$$

A rise from about 0.28 cents per 10 cent resistor to 0.90 cents per 10 cent resistor. If you believe the quadratic loss function, the mean shift would wipe out the advantage gained by reducing the variance.

Compare this with the result of the mean shift in a three sigma process.

Create a Monte Carlo simulation of the manufacture of the resistors.

n := 1000 Number of resistors to be manufactured

$$\sigma := \frac{\Delta_0}{3}$$

$$\mu := m + 1.5 \cdot \sigma$$

$R := \text{rnorm}(n, \mu, \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

$$\text{number_of_bins} := 20$$

$$\text{width_of_bins} := \frac{4 \cdot \Delta_0}{\text{number_of_bins}}$$

$$j := 1.. \text{number_of_bins} + 1$$

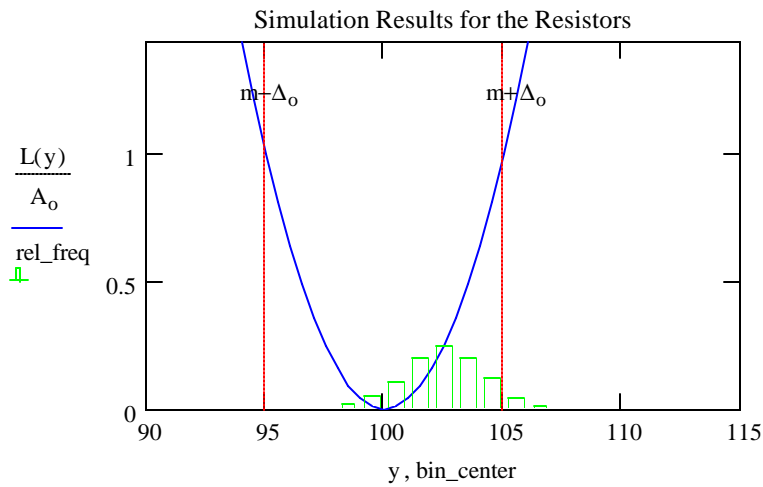
$$\text{bin}_j := m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$$

Define a vector with the start and end points of the bins.

$$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n}$$

Compute the relative frequency distribution over interval.

$$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 3.513 in cents

How does this compare to the theoretically derived figure?

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 3.611 \quad \text{Or, according to Phadke 2.10.}$$

Both the three sigma process and the six sigma process roughly tripled in quality loss due to the mean shift. I'd say they are equally sensitive to the mean shift although I realize the magnitude of the effect on the three sigma process was greater.

d) Consider the case of a really poor process

Create a Monte Carlo simulation of the manufacture of the resistors.

$n := 1000$ Number of resistors to be manufactured

$$\sigma := \frac{\Delta_o}{2}$$

$$\mu := m + \Delta_o$$

$R := \text{rnorm}(n, \mu, \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

$\text{number_of_bins} := 20$

$$\text{width_of_bins} := \frac{4 \cdot \Delta_o}{\text{number_of_bins}}$$

$j := 1.. \text{number_of_bins} + 1$

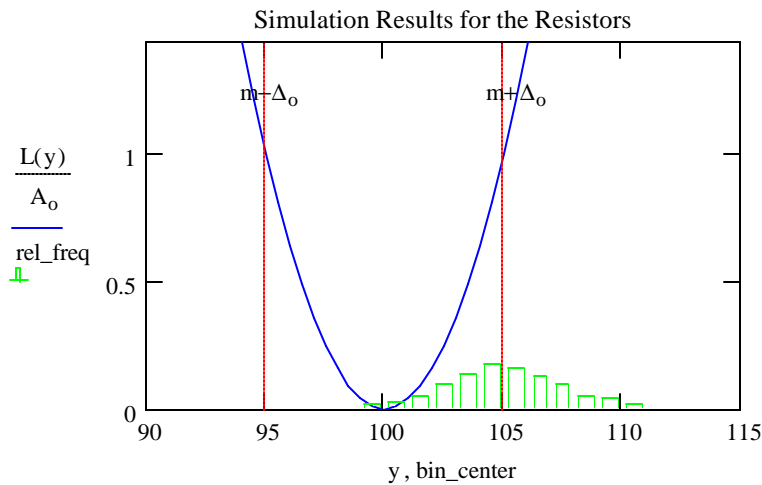
$$\text{bin}_j := m - 2 \cdot \Delta_o + \text{width_of_bins} \cdot j$$

Define a vector with the start and end points of the bins.

$$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n}$$

Compute the relative frequency distribution over interval.

$$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 12.698 in cents

How does this compare to the theoretically derived figure?

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 12.5 \quad \text{Or, according to Phadke 2.10.}$$

The expected loss of 12 cents is greater than the cost of the resistors themselves at 10 cents. This seems rather odd since very nearly half of the resistors meet the tolerance. If inspection were cheap and you really can sell half of your resistors as 5% resistors, then shouldn't the quality loss be more like 5 cents?

The outliers at 2 times the tolerance width are penalized with a loss function of 4 times the cost of the product. Does that seem reasonable? Perhaps in such cases of extreme variation, the quality loss function should be modified. What if the quality loss remained constant at A_0 everywhere outside the tolerance band.

In fairness, this is an extreme example. The quadratic loss function seems to work well in a wide range of industrial settings.

e) Consider the effect of a change to a uniform distribution.

Create a Monte Carlo simulation of the manufacture of the resistors.

$n := 1000$ Number of resistors to be manufactured

$\sigma := \frac{\Delta_0}{3}$ $\sigma = 1.667$

$\mu := m$

$R := \text{runif}(n, \mu - \sqrt{3} \cdot \sigma, \mu + \sqrt{3} \cdot \sigma)$ Create a vector with all of the resistance values of the resistors we manufactured.

$\text{Stdev}(R) = 1.649$ as you can see, a width of 2 root 3 sigma works

Set up the format for a histogram of the data.

$\text{number_of_bins} := 20$

$\text{width_of_bins} := \frac{4 \cdot \Delta_0}{\text{number_of_bins}}$

$j := 1.. \text{number_of_bins} + 1$

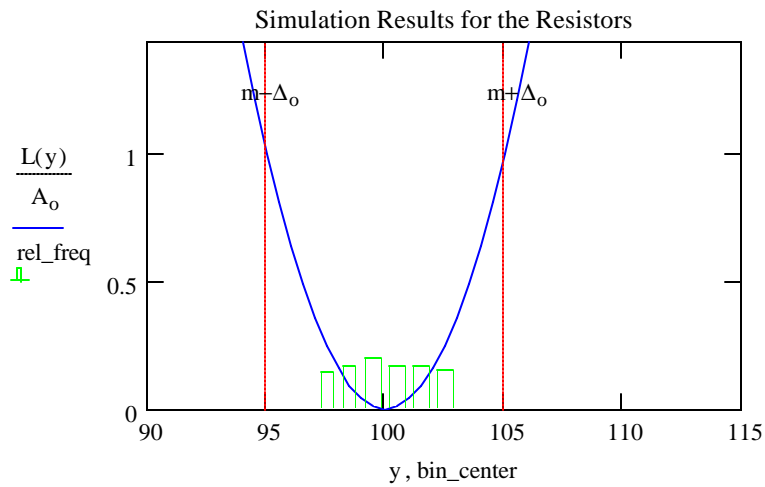
$\text{bin}_j := m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$

Define a vector with the start and end points of the bins.

$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n}$

Compute the relative frequency distribution over interval.

$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 1.087 in cents

How does this compare to the theoretically derived figure?

$$k_1 \cdot [(\mu - m)^2 + \sigma^2] = 1.111 \quad \text{Or, according to Phadke 2.10.}$$

Same answer as in (a). The shape of the distribution just doesn't matter!

f) Consider the effect of an asymmetric loss function.

Solution:

$$k_1 := 2 \cdot \frac{A_0}{\Delta_0^2} \quad k_2 := \frac{A_0}{\Delta_0^2}$$

$$L(y) := \begin{cases} k_1 \cdot (y - m)^2 & \text{if } y > m \\ k_2 \cdot (y - m)^2 & \text{if } y \leq m \end{cases} \quad \text{Define the quadratic loss function}$$

$$y := m - 1.2 \cdot \Delta_0, m - 1.2 \cdot \Delta_0 + \frac{\Delta_0}{10} .. m + 1.2 \cdot \Delta_0 \quad \text{Define a range over } y \text{ for the purpose of plotting}$$

Create a Monte Carlo simulation of the manufacture of the resistors.

$$n := 1000 \quad \text{Number of resistors to be manufactured}$$

$$\sigma := \frac{\Delta_0}{3}$$

$$\mu := m + 0 \cdot \sigma$$

$$R := \text{rnorm}(n, \mu, \sigma) \quad \text{Create a vector with all of the resistance values of the resistors we manufactured.}$$

Set up the format for a histogram of the data.

$$\text{number_of_bins} := 20$$

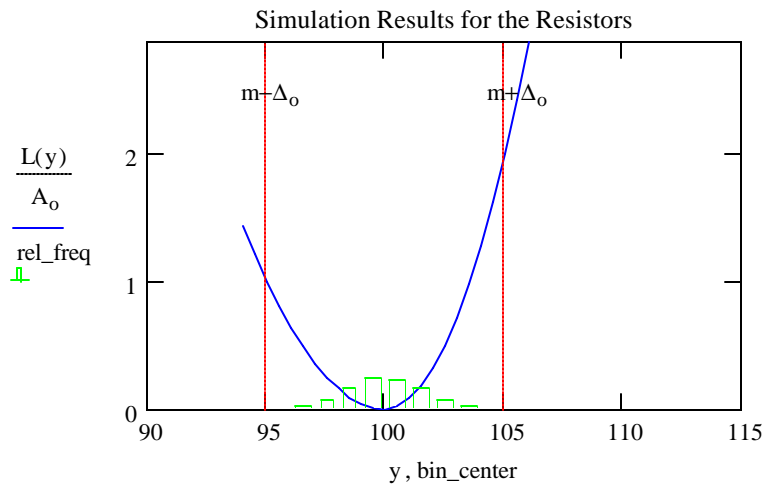
$$\text{width_of_bins} := \frac{4 \cdot \Delta_0}{\text{number_of_bins}}$$

$$j := 1.. \text{number_of_bins} + 1$$

$$\text{bin}_j := m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j \quad \text{Define a vector with the start and end points of the bins.}$$

$$\text{rel_freq} := \frac{\text{hist}(\text{bin}, R)}{n} \quad \text{Compute the relative frequency distribution over interval.}$$

$$\text{bin_center} := \text{bin} + 0.5 \cdot \text{width_of_bins}$$



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 1.541 in cents

The quality loss rose compared to the figure in (a) as we might expect. Looking at the histogram above, doesn't it seem we'd be better off aiming for slightly below the center of the range? Let's check.

Create a Monte Carlo simulation of the manufacture of the resistors.

`n := 10000` Number of resistors to be manufactured

$$\sigma := \frac{\Delta_0}{3}$$

$$\mu := m - 0.1 \cdot \Delta_0$$

`R := rnorm(n, μ, σ)` Create a vector with all of the resistance values of the resistors we manufactured.

Set up the format for a histogram of the data.

number_of_bins := 20

width_of_bins := $\frac{4 \cdot \Delta_0}{\text{number_of_bins}}$

j := 1.. number_of_bins + 1

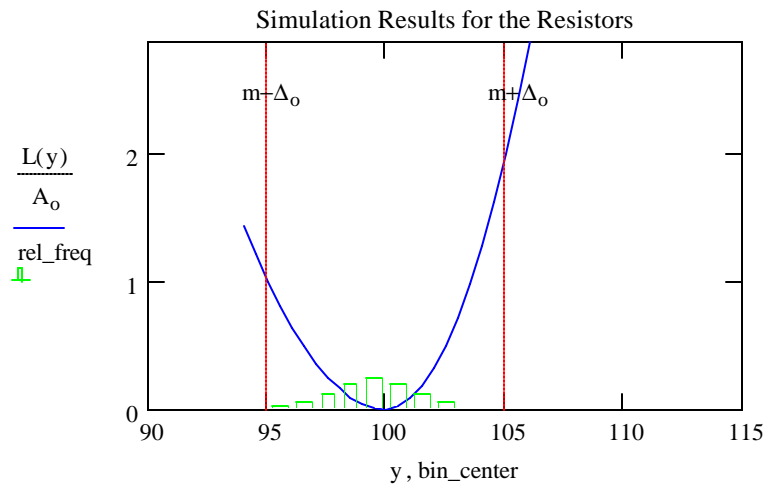
bin_j := $m - 2 \cdot \Delta_0 + \text{width_of_bins} \cdot j$

Define a vector with the start and end points of the bins.

rel_freq := $\frac{\text{hist}(\text{bin}, R)}{n}$

Compute the relative frequency distribution over interval.

bin_center := bin + 0.5 · width_of_bins



$$\text{Average_quality_loss} := \frac{1}{n} \cdot \sum_{i=1}^n L(R_i)$$

Average_quality_loss = 1.568 in cents

The quality loss is lower by aiming for off center. But the improvement isn't very substantial. Our efforts are better spend reducing variation and staying on target! Luckily, that is the focus of the majority of this course.