Electromagnetic Formation Flight

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OVERVIEW

Description of the Effort

The Massachusetts Institute of Technology Space Systems Lab (MIT SSL) and the Lockheed Martin Advanced Technology Center (ATC) are collaborating to explore the potential for an Electro-Magnetic Formation Flight (EMFF) system applicable to Earth-orbiting satellites flying in close formation.

Progress Overview

At MIT, work on EMFF has been pursued on two fronts: the MIT conceive, design, implement and operate (CDIO) class, and the MIT SSL research group.

This report summarizes recent progress made in the MIT SSL research group with regards to analyzing the dynamics of a two-spacecraft electromagnetic formation flying system. It is a follow-up to the work presented in the October 2002 progress report, which introduced the nonlinear and linearized dynamics of such a system. In this report, we summarize our recent findings about the controllability and stability of this system.

1. Introduction

In this report, we sumarize the dynamics of the electromagnetic formation flying (EMFF) system of spacecraft introduced in the October 2002 progress report [1], and then continue by presenting results from recent controllability and stability analyses performed on similar EMFF systems. In particular, we consider a formation of two spacecraft under no external forces and rotating about a common origin. Each spacecraft contains a specified configuration of fixed electromagnets (EM) and reaction wheels (RW) for use as position and attitude actuators.

We begin in Section 2 by reviewing the results presented in [1]. **In Sections 3 and 4, respectively, we define and respond to certain questions pertaining to the controllability and stability of the system at hand.** We then summarize our results and draw conclusions in Section 5.

2. Background

We begin by reviewing the linearized dynamics of a system similar to those considered in this exercise. Reference [1] describes the dynamics of a two-spacecraft EMFF system, where each spacecraft has a single fixed electromagnet, nominally pointed along the line of sight between the two spacecraft. Each spacecraft also has a single reaction wheel, nominally oriented perpendicular to the plane of rotation of the two spacecraft. Such a system has nine degrees of freedom (or eighteen state variables):

$$\boldsymbol{x} = \begin{bmatrix} \Delta r\ \Delta\phi\ \Delta\psi\ \Delta\alpha_1\ \Delta\alpha_2\ \Delta\alpha_3\ \Delta\beta_1\ \Delta\beta_2\ \Delta\beta_3 \end{bmatrix}^{\mathrm{T}} \qquad (2.1)$$

where $\Delta r, \Delta\phi, \Delta\psi$ are the relative displacements of the vehicles in curvilinear coordinates depicted in Figures 1 and 2, $\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3$ are the Euler angles of the first spacecraft (denoted spacecraft "A") about its body-fixed z, y, and x axes, respectively, and $\Delta\beta_1, \Delta\beta_2, \Delta\beta_3$ are the corresponding Euler angles of the second spacecraft (denoted spacecraft "B") about its body-fixed coordinate axes.
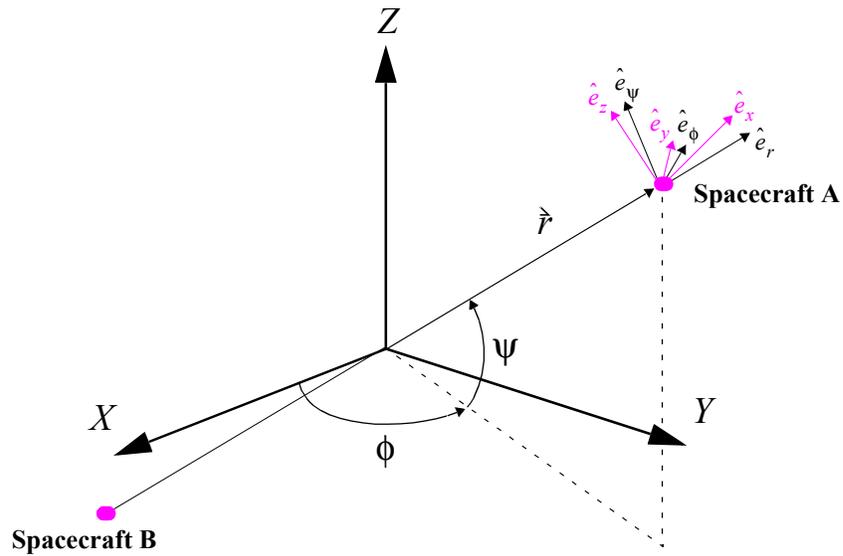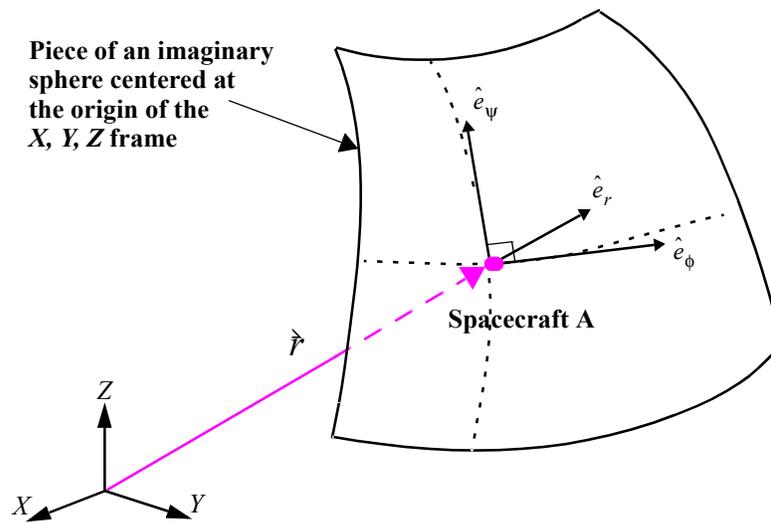
Figure 1 Geometry of Two-Spacecraft Array

Figure 2 Local Curvilinear Coordinate Frame on Spacecraft A

The linearized equations of motion for this system were presented in second-order form in [1] as:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{zz,s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{zz,s} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w}
\end{bmatrix}
\begin{Bmatrix}
\Delta\ddot{r} \\
\Delta\ddot{\phi} \\
\Delta\ddot{\psi} \\
\Delta\ddot{\alpha}_1 \\
\Delta\ddot{\alpha}_2 \\
\Delta\ddot{\alpha}_3 \\
\Delta\ddot{\beta}_1 \\
\Delta\ddot{\beta}_2 \\
\Delta\ddot{\beta}_3
\end{Bmatrix}
$$

$$
+
\begin{bmatrix}
0 & -2r_0\dot{\phi}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\dot{\phi}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & mr_0^2\dot{\phi}_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -mr_0^2\dot{\phi}_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & mr_0^2\dot{\phi}_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -mr_0^2\dot{\phi}_0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta\dot{r} \\
\Delta\dot{\phi} \\
\Delta\dot{\psi} \\
\Delta\dot{\alpha}_1 \\
\Delta\dot{\alpha}_2 \\
\Delta\dot{\alpha}_3 \\
\Delta\dot{\beta}_1 \\
\Delta\dot{\beta}_2 \\
\Delta\dot{\beta}_3
\end{Bmatrix}
$$

(2.2)

$$
+
\begin{bmatrix}
-5\dot{\phi}_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & r_0\dot{\phi}_0^2 & 0 & -c_1 & 0 & 0 & -c_1 & 0 \\
0 & 0 & 0 & -2c_0 & 0 & 0 & -c_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2c_0 & 0 & 0 & -c_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -c_0 & 0 & 0 & -2c_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -c_0 & 0 & 0 & -2c_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta r \\
\Delta\phi \\
\Delta\psi \\
\Delta\alpha_1 \\
\Delta\alpha_2 \\
\Delta\alpha_3 \\
\Delta\beta_1 \\
\Delta\beta_2 \\
\Delta\beta_3
\end{Bmatrix}
=
\begin{bmatrix}
c_2 & c_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & K_T & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta\mu_A \\
\Delta\mu_B \\
\Delta i_{RW,A} \\
\Delta i_{RW,B}
\end{Bmatrix}
$$

where:

$$
c_0 \equiv \frac{-mr_0^2\dot{\phi}_0^2}{3} , \quad c_1 \equiv \frac{-r_0\dot{\phi}_0^2}{2} , \text{ and } c_2 \equiv -\dot{\phi}_0\sqrt{\frac{3\mu_0}{32\pi mr_0^3}}
$$

(2.3)

K_T is the reaction wheel torque constant, and all remaining values are defined in [1].

We recognize that, rather than deriving the linearized dynamics of the similar systems considered in the following sections from first principles, we can simply modify Equation 2.2 by removing the appropriate degrees of freedom from the dynamic matrices and altering the actuator coefficient matrix as necessary.

3. Controllability Analysis

We begin by considering the linearized dynamics of two vehicles *in two dimensions* (a plane), where the total angular momentum of the system is zero (wheels plus spacecraft), but the rotational angular momentum of the two vehicles about a common origin is non-zero.

Part a)

In this case, each spacecraft has a single dipole, fixed to the spacecraft and nominally pointed along the line of sight to the other spacecraft. Each spacecraft also has a single reaction wheel, oriented perpendicular to the plane of rotation of the spacecraft.

Also, we assume the vehicles are free to rotate about any axis passing through their center of mass (any "central" axis), and are not necessarily constrained to rotate about axes perpendicular to the plane of rotation. Hence each spacecraft has three rotational degrees of freedom. After analyzing this case, and before moving on to Part b, we will also consider a simplified version, in which rotations are constrained to axes perpendicular to the system's rotational plane, and each spacecraft thus has only one rotational degree of freedom.

With these assumptions, we recognize that the only difference between this system and the system in [1] is that in the present system, the two spacecraft do not leave the nominal horizontal plane of rotation (the *X-Y* plane in Figure 1). Therefore, the system does not have a degree of freedom associated with the latitude, $\Delta\psi$, and we modify Equation 2.2 by striking the third column of each square "coefficient" matrix, as well as the third row of the entire matrix equation. This results in the following modified equations of motion:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{zz,s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{zz,s} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{rr,s}+I_{rr,w}
\end{bmatrix}
\begin{Bmatrix}
\Delta\ddot{r} \\
\Delta\ddot{\phi} \\
\Delta\ddot{\alpha}_1 \\
\Delta\ddot{\alpha}_2 \\
\Delta\ddot{\alpha}_3 \\
\Delta\ddot{\beta}_1 \\
\Delta\ddot{\beta}_2 \\
\Delta\ddot{\beta}_3
\end{Bmatrix}
$$

$$
+
\begin{bmatrix}
0 & -2r_0\dot{\phi}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\dot{\phi}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & mr_0^2\dot{\phi}_0 & 0 & 0 & 0 \\
0 & 0 & 0 & -mr_0^2\dot{\phi}_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & mr_0^2\dot{\phi}_0 \\
0 & 0 & 0 & 0 & 0 & 0 & -mr_0^2\dot{\phi}_0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta\dot{r} \\
\Delta\dot{\phi} \\
\Delta\dot{\alpha}_1 \\
\Delta\dot{\alpha}_2 \\
\Delta\dot{\alpha}_3 \\
\Delta\dot{\beta}_1 \\
\Delta\dot{\beta}_2 \\
\Delta\dot{\beta}_3
\end{Bmatrix}
\tag{3.1}
$$

$$
+
\begin{bmatrix}
-5\dot{\phi}_0^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_1 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & -2c_0 & 0 & 0 & -c_0 & 0 & 0 \\
0 & 0 & 0 & -2c_0 & 0 & 0 & -c_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_0 & 0 & 0 & -2c_0 & 0 & 0 \\
0 & 0 & 0 & -c_0 & 0 & 0 & -2c_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta r \\
\Delta\phi \\
\Delta\alpha_1 \\
\Delta\alpha_2 \\
\Delta\alpha_3 \\
\Delta\beta_1 \\
\Delta\beta_2 \\
\Delta\beta_3
\end{Bmatrix}
=
\begin{bmatrix}
c_2 & c_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & K_T & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\Delta\mu_A \\
\Delta\mu_B \\
\Delta i_{RW,A} \\
\Delta i_{RW,B}
\end{Bmatrix}
$$

Equation 3.1 is in second order form:

$$
M\ddot{x} + C\dot{x} + Kx = Fu \tag{3.2}
$$

where:

$$
x = \begin{bmatrix} \Delta r & \Delta\phi & \Delta\alpha_1 & \Delta\alpha_2 & \Delta\alpha_3 & \Delta\beta_1 & \Delta\beta_2 & \Delta\beta_3 \end{bmatrix}^{\mathrm{T}} \tag{3.3}
$$

$$
u = \begin{bmatrix} \Delta\mu_A & \Delta\mu_B & \Delta i_{RW,A} & \Delta i_{RW,B} \end{bmatrix}^{\mathrm{T}} \tag{3.4}
$$

and M, C, K, and F are the appropriate matrix coefficients. The control vector, $\boldsymbol{u}$, contains four control variables: $\Delta\mu_A$ and $\Delta\mu_B$, which represent the deviation of the EM magnetic dipole moments on spacecraft A and B, respectively, from their nominal, steady-state values, and $\Delta i_{RW,A}$ and $\Delta i_{RW,B}$, which represent the deviation of the RW motor currents on spacecraft A and B, respectively, from their nominal values.

To investigate the controllability of this system using linear control analysis tools, we recast Equation 3.1 in first order (state-space) form as:

$$\dot{\tilde{\boldsymbol{x}}} = A\tilde{\boldsymbol{x}} + B\boldsymbol{u} \tag{3.5}$$

where:

$$\tilde{\boldsymbol{x}} = [\ \boldsymbol{x}\ \ \dot{\boldsymbol{x}}\ ]^{\mathrm{T}} \tag{3.6}$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{3.7}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} \tag{3.8}$$

I represents an $\frac{n}{2} \times \frac{n}{2}$ identity matrix, where $\frac{n}{2}$ is the number of degrees of freedom of the system (and n is the number of states).

The system described in Equations 3.3-3.8 is represented by eight degrees of freedom, or 16 state variables. This is expected, since we have removed one degree of freedom ($\Delta\psi$) from the system described in [1].

To assess the controllability of the linearized dynamics in 3.5, we form the "controllability matrix," defined as (See [4].):

$$C = \left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array} \right] \tag{3.9}$$

From linear control theory, we know that the system is only controllable if $\text{rank}(C) = n$.

Substituting A and B as defined by 3.7-3.8 and 3.1 into 3.9, we obtain the controllability matrix. Testing the rank using the Matlab "rank" command yields:

$$\text{rank}(C) = 8 < n \tag{3.10}$$

so that clearly the system at hand is not fully controllable. **Since** $\text{rank}(C) = 8$ **in this case, we see that only eight states (or four degrees of freedom) are controllable**, and this system as a whole is *not* fully controllable.

This result makes sense intuitively, since we have *eight degrees of freedom* (two translational degrees of freedom, Δr and $\Delta\phi$, and six rotational degrees of freedom, $\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3, \Delta\beta_1, \Delta\beta_2$, and $\Delta\beta_3$), *but only four actuators* (one EM and one RW on each spacecraft).

It is also interesting to consider a simplified version of this geometry, in which the system has only four degrees of freedom, Δr, $\Delta\phi$ $\Delta\alpha_1$, and $\Delta\beta_1$, and thus eight state variables. In this case, the bodies are allowed to translate within a plane as before, but are now constrained to rotate only about axes perpendicular to the system's plane of rotation. We modify the dynamic equations in 3.1 by striking the appropriate rows and columns from the matrices M, C, K, and F, as well as from the state vector:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_0 & 0 & 0 \\ 0 & 0 & I_{zz,s} & 0 \\ 0 & 0 & 0 & I_{zz,s} \end{bmatrix} \begin{Bmatrix} \Delta\ddot{r} \\ \Delta\ddot{\phi} \\ \Delta\ddot{\alpha}_1 \\ \Delta\ddot{\beta}_1 \end{Bmatrix} + \begin{bmatrix} 0 & -2r_0\dot{\phi}_0 & 0 & 0 \\ 2\dot{\phi}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\alpha}_1 \\ \Delta\dot{\beta}_1 \end{Bmatrix} + \begin{bmatrix} -5\dot{\phi}_0^2 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_1 \\ 0 & 0 & -2c_0 & -c_0 \\ 0 & 0 & -c_0 & -2c_0 \end{bmatrix} \begin{Bmatrix} \Delta r \\ \Delta\phi \\ \Delta\alpha_1 \\ \Delta\beta_1 \end{Bmatrix}
$$

$$
= \begin{bmatrix} c_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K_T & 0 \\ 0 & 0 & 0 & K_T \end{bmatrix} \begin{Bmatrix} \Delta\mu_A \\ \Delta\mu_B \\ \Delta i_{RW,A} \\ \Delta i_{RW,B} \end{Bmatrix}
$$

$$(3.11)$$

We then assemble the A and B matrices using 3.7 and 3.8 and form the controllability matrix using 3.9. Finally, the rank test of the controllability matrix yields:

$$
\text{rank}(C) = 8 = n \tag{3.12}
$$

Hence this simplified system is, indeed, fully controllable. This result is quite interesting: it tells us that we can fully control a two-spacecraft "planar" system (with degrees of freedom Δr, $\Delta\phi$, $\Delta\alpha_1$, and $\Delta\beta_1$) using only four actuators (one EM and one RW on each spacecraft).

Part b)

The geometry in this case is similar to the more complicated system in Part a, with the exception that each spacecraft has one additional electromagnet, oriented perpendicular to the first and still in the plane of rotation of the two spacecraft.

*Since the system geometry is identical to that in Part a, the non-actuated dynamics (and hence the A matrix) from 3.1 remain the same. However, because the actuator configuration has changed, the control vector, **u**, and the linearized actuator coefficient matrix, B, will change.* We define the new control vector as:

$$
\boldsymbol{u} = \begin{bmatrix} \Delta\mu_{A1} & \Delta\mu_{A2} & \Delta\mu_{B1} & \Delta\mu_{B2} & \Delta i_{RW,A} & \Delta i_{RW,B} \end{bmatrix}^{\text{T}} \tag{3.13}
$$

where $\Delta\mu_{A1}$ and $\Delta\mu_{B1}$ represent the original EM dipole moments from Part a, and $\Delta\mu_{A2}$ and $\Delta\mu_{B2}$ represent the dipole moments of the new orthogonal EMs on each spacecraft.

With two orthogonal electromagnets on each spacecraft, the net electromagnetic force on spacecraft A may now be represented as a sum of the interactions between the individual electromagnets on A and B:

$$F_{A/B} = F_{A1/B1} + F_{A1/B2} + F_{A2/B1} + F_{A2/B2} = -F_{B/A} \qquad (3.14)$$

where $F_{Ai/Bj}$ ($i = 1, 2$ and $j = 1, 2$) represents the force on the ith EM on spacecraft A due to the jth EM on spacecraft B. Note that the force on B is equal in magnitude and opposite in direction to that on A.

Similar expressions yield the torque on each spacecraft due to the electromagnetic interactions with the other spacecraft:

$$T_{A/B} = T_{A1/B1} + T_{A1/B2} + T_{A2/B1} + T_{A2/B2} \qquad (3.15)$$

$$T_{B/A} = T_{B1/A1} + T_{B1/A2} + T_{B2/A1} + T_{B2/A2} \qquad (3.16)$$

Unlike the forces, the torques on A and B are *not* equal in magnitude and opposite in direction.

The resulting nonlinear force and torque expressions may be written as functions of the EM dipole strengths, the separation distance between the spacecraft, and the Euler-angle attitude representations of the spacecraft, as described in detail in [1]. This yields complicated expressions for the resultant forces and torques. These expressions are calculated using the "symbolic toolbox" in Matlab.

In order to append these new forces and torques into our linearized dynamics, we must linearize them about nominal values. The nominal separation distance is r_0, and the nominal Euler angles are zero, as in previous linearizations. Also, the line-of-sight-EM dipole

moments on A and B are linearized about the same nominal values as before, in order to provide the necessary centripetal acceleration for nominal steady-state spin of the system:

$$\mu_{A1,0} = \mu_{B1,0} = \sqrt{\frac{32\pi m r_0^5 \dot{\phi}_0^2}{3\mu_0}} \tag{3.17}$$

so that:

$$\mu_{A1} = \mu_{A1,0} + \Delta\mu_{A1}, \quad \mu_{B1} = \mu_{B1,0} + \Delta\mu_{B1} \tag{3.18}$$

Hence we have only to specify nominal values for the "new" EMs (orthogonal to the line-of-sight EMs) on spacecraft A and B. In order to avoid nonhomogeneous forces and torques due to interactions between the nominal magnetic moments of the new EMs and those of the line-of-sight EMs, we must linearize the new EMs about zero. Hence:

$$\mu_{A2,0} = \mu_{B2,0} = 0 \tag{3.19}$$

so that:

$$\mu_{A2} = \Delta\mu_{A2}, \quad \mu_{B2} = \Delta\mu_{B2} \tag{3.20}$$

The new actuator coefficient matrix, F, resulting from the linearized forces and torques is:

$$F = \begin{bmatrix} c_2 & 0 & c_2 & 0 & 0 & 0 \\ 0 & \dfrac{-c_2}{2} & 0 & \dfrac{-c_2}{2} & 0 & 0 \\ 0 & \dfrac{2mr_0c_2}{3} & 0 & \dfrac{mr_0c_2}{3} & K_T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{mr_0c_2}{3} & 0 & \dfrac{2mr_0c_2}{3} & 0 & K_T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.21}$$

where the six columns correspond to the six control variables in Equation 3.13.

Note that the dynamic matrices, *M*, *C*, and *K*, are unchanged from Part a, since only the actuator configuration has changed. Hence the *A* matrix is also unchanged. We proceed by recalculating *B* using Equation 3.8, and substituting the new *F* matrix from 3.21. We then form the controllability matrix for this configuration using Equation 3.9, and using the Matlab "rank" command, we find:

$$\text{rank}(C) = 8 < n \tag{3.22}$$

where we recall that $n = 16$ for this geometry. **Hence, again, only four degrees of freedom (or eight state variables) are controllable in this case, so this actuator configuration does not improve upon that of Part a.**

Finally, we consider adding two additional reaction wheels to each spacecraft, with spin-axes fixed in the plane of rotation and orthogonal to each other and to the existing RW. In this case, the control vector is:

$$\boldsymbol{u} = \begin{bmatrix} \Delta\mu_{A1} & \Delta\mu_{A2} & \Delta\mu_{B1} & \Delta\mu_{B2} & \Delta i_{RW,A1} & \Delta i_{RW,A2} & \Delta i_{RW,A3} & \Delta i_{RW,B1} & \Delta i_{RW,B2} & \Delta i_{RW,B3} \end{bmatrix}^{\text{T}} \tag{3.23}$$

and the linearized actuator coefficient matrix is:

$$F = \begin{bmatrix} c_2 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-c_2}{2} & 0 & \dfrac{-c_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{2mr_0c_2}{3} & 0 & \dfrac{mr_0c_2}{3} & K_T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_T & 0 & 0 & 0 \\ 0 & \dfrac{mr_0c_2}{3} & 0 & \dfrac{2mr_0c_2}{3} & 0 & 0 & 0 & K_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_T \end{bmatrix} \tag{3.24}$$

Substituting 3.24 into 3.8 to solve for *B*, then forming the controllability matrix and testing the rank, we find:

$$\text{rank}(C) \ = \ 16 \ = \ n \tag{3.25}$$

This is an important result; it tells us that if each spacecraft in two dimensions has two orthogonal EMs within the plane of rotation, as well as three orthogonal RWs (two in the plane of rotation and one orthogonal to that plane), then the system will be fully controllable in two dimensions. Note that this includes not only translation within the plane and rotation about axes orthogonal to the plane, but also rotation about arbitrary axes passing through the spacecraft center of mass!

A useful extension of this will be to check the controllability of such a system *in three dimensions*. A reasonable conjecture is that such a system will be uncontrollable, but if a third EM is added to each spacecraft and oriented orthogonal to the plane of rotation, then the system may be fully controllable. This conjecture will be investigated in the near future.

4. Stability Analysis

We now consider the linearized dynamics of two spacecraft in three dimensions, rotating about a common origin as before (in a plane), with a total system angular momentum of zero as before (spacecraft plus reaction wheels). We will examine the linearized open-loop dynamics of this system.

Part a)

We are now working in three dimensions, so we return to the configuration in reference [1]. The *A* matrix is formed using Equation 3.7, based on the three-dimensional dynamics in Equation 2.2. Using the "eig" command in the Matlab symbolic toolbox with the *A* matrix as a function input, **we find the eigenvalues of this system to be**:

$$\lambda_{1,2,3,4,5,6} = 0 \tag{4.1}$$

$$\lambda_{7,8} = \pm\dot{\phi}_0 \tag{4.2}$$

$$\lambda_{9,10} = \pm i\dot{\phi}_0 \tag{4.3}$$

$$\lambda_{11,12} = \pm i\frac{r_0\dot{\phi}_0}{(I_{rr,s}+I_{rr,w})}\sqrt{m\left(mr_0^2 + \frac{I_{rr,s}+I_{rr,w}}{3}\right)} \tag{4.4}$$

$$\lambda_{13,14} = \pm i\frac{r_0\dot{\phi}_0}{(I_{rr,s}+I_{rr,w})}\sqrt{m(mr_0^2 + I_{rr,s}+I_{rr,w})} \tag{4.5}$$

$$\lambda_{15,16} = \pm i r_0\dot{\phi}_0\sqrt{\frac{m}{3I_{zz,s}}} \tag{4.6}$$

$$\lambda_{17,18} = \pm i r_0\dot{\phi}_0\sqrt{\frac{m}{I_{zz,s}}} \tag{4.7}$$

A pole-zero map based on sample geometric and mass values is shown in Figure 3.

Part b)

We now compare the poles of this system to the poles of the 1-dimensional airtrack system used in the CDIO class to demonstrate EM control. Please see references [2] and [5] for a detailed description of the airtrack hardware, dynamics, and experimental results.

From reference [5], we recall that the poles of the 1-D airtrack system *in its unstable configuration* are:

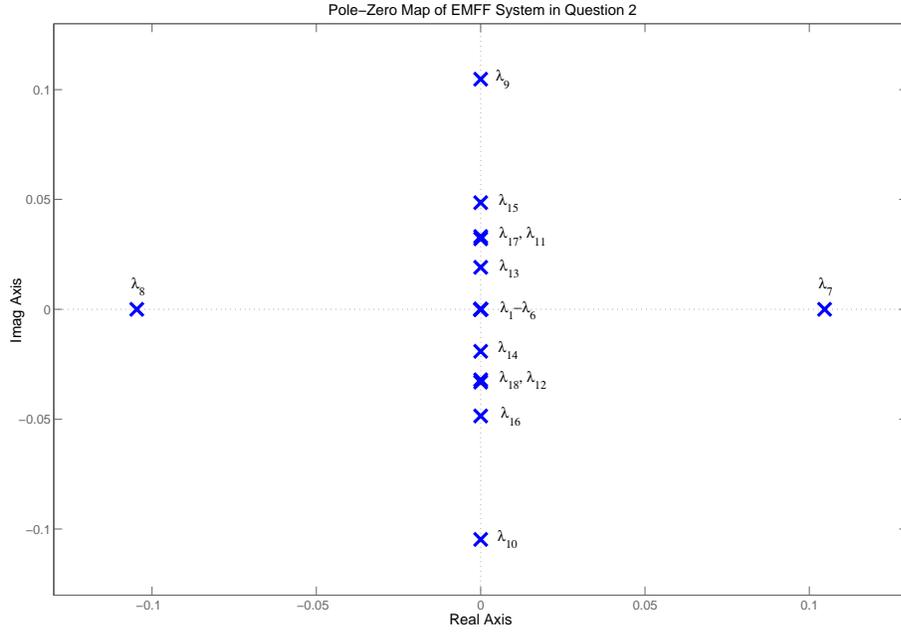$$\lambda_{1,2}^{\text{Unstable}} = \pm\sqrt{\frac{4c\mu^2}{mr_0^5}} \tag{4.8}$$

Figure 3 Pole-Zero Map for System in Question 2, Using Sample Geometric Values

where μ^2 is the product of magnetic dipole moments of the two EMs, r_0 is their nominal separation distance, m is the mass of the sliding (non-fixed) EM, and c is a constant:

$$c = \frac{3\mu_0}{2\pi} \tag{4.9}$$

By comparing these poles with the those of our three-dimensional formation flying system, we find that the airtrack poles are of the same form as λ_7 and λ_8 in Equation 4.2. In both cases, the poles are real, one positive and one negative, and mirror each other about the imaginary axis in the complex plane as shown in Figure 3.

In the airtrack case, the pair of real poles is expected. We know the configuration to be inherently unstable, as described in [5], and this is confirmed by the fact that one of the system poles is in the right-half complex plane. As discussed in [5], the eigenmode corresponding to this eigenvalue represents a divergence of the dynamics; in particular, it represents the fact that the "sliding" magnet tends to fall away from the fixed magnet if the

attractive force is slightly too weak, but to accelerate toward the fixed magnet if the force is slightly too strong. We see the same unstable physics occurring in the formation flying system. The attractive force between the two magnets maintains the centripetal acceleration necessary for the system to spin; if the force is slightly too weak, the magnets will quickly "fall" away from one another. If, however, the force is slightly too strong, the magnets will accelerate toward each other. Hence we see very similar physics between the two situations, and our intuition is confirmed by the fact that the system poles are of the same mathematical form.

Next, we consider the poles of the airtrack system *in its neutrally stable configuration*:

$$\lambda_{1,2}^{\text{Stable}} = \pm i \sqrt{\frac{4c\mu^2}{mr_0^5}} \tag{4.10}$$

These are of the same form as λ_9 and λ_{10} in Equation 4.3. In both cases, there are two imaginary poles, one positive and one negative, that mirror each other about the real axis in the complex plane as shown in Figure 3.

In this configuration, we know the linear airtrack to be neutrally stable; once the sliding magnet is perturbed, it will oscillate indefinitely with respect to the fixed magnet. (In reality, this configuration is stable because the small amount of friction between the sliding magnet and the track adds damping to the system and moves the poles slightly into the left-half complex plane.) Since the formation flying system has poles of the same form, it must have a corresponding neutrally stable eigenmode. By studying the eigenvectors produced in Matlab, we see that the ψ and $\dot{\psi}$ components of this eigenvector are 90 degrees out of phase from one another, indicating that this mode corresponds to a sinusoidal "tilting" of the entire plane of rotation of the two spacecraft. This tilting occurs about an axis located in the global *X-Y* plane and passing through the origin of the global frame.

One encouraging aspect of the pole-zero map in Figure 3 is that only one pole is strictly unstable. Hence if we can design a controller that will stabilize this mode, we have a good chance of controlling the entire system.

Part c)

As a final exercise, we investigate how the formation flying system's poles in Equations 4.1-4.7 change as a function of the angular momentum of the vehicles about the origin of rotation.

First we recognize that the first six poles, $\lambda_1 - \lambda_6$, are equal to zero and thus are not affected by the angular momentum of the spinning vehicles or any other geometric properties of the system. These poles represent the six rigid-body modes of the system.

Next, we see that λ_7 - λ_{10} are directly proportional to the nominal spin rate, $\dot{\phi}_0$, of the system. **Hence as the nominal spin rate varies from zero to infinity, these four poles move out from the origin, each along a different axis of the complex plane. Since the positive-real eigenvalue, λ_7, moves further into the right-half complex plane as the nominal spin rate increases, we see that the system becomes increasingly unstable. Hence the larger the angular momentum of the vehicles spinning about the origin, the more unstable the system will be.**

All the remaining poles, both real and imaginary, grow in magnitude with increasing $\dot{\phi}_0$, *m*, and r_0. Hence as the angular momentum, mass, and separation distance of the space-craft increase, these poles move further out from the origin toward infinity. Note that since all the arguments of the radicals in 4.1-4.7 are necessarily positive, there is no combination of values that can change the sign of any argument, so that it is not possible for real poles to become complex or imaginary for certain geometries, or vice versa. The result is that **as $\dot{\phi}_0$, *m*, or r_0 increase, the poles $\lambda_{11} - \lambda_{18}$ move toward infinity along the axes on which they lie**, so that the positive-real poles move further to the right along the positive-real axis, and so forth. This makes sense, since we expect the system to

become increasingly unstable as the angular momentum, mass, and separation distance increase.

5. Conclusions

The results presented in this report are summarized here.

- In Section 3, Part a, we discovered that if we retain all of the rotational degrees of freedom of each spacecraft in the two-dimensional configuration, only four of the eight system degrees of freedom will be controllable. However, if we eliminate spacecraft rotations about axes within the system's plane of rotation, then we have only four degrees of freedom, all of which are controllable.

- In Section 3, Part b, we find with only one RW (oriented perpendicular to the plane of rotation) that only four of the eight degrees of freedom are controllable. If, however, we append two RWs per spacecraft (perpendicular to each other and lying in the system's plane of rotation), then all eight degrees of freedom of the system will be controllable.

- In Section 4, Part a, we determined the 18 eigenvalues of the three-dimensional system. We assumed sample mass and geometric properties and plotted the resulting numerical values of the poles in the complex plane. Six poles lie at the origin, two poles lie on the real axis, and ten poles lie on the imaginary axis.

- In Section 4, Part b, we compare the system's poles to those of the airtrack used for the CDIO class. We found two of the EMFF system's poles were similar in form to the airtrack's poles in its *stable* configuration, and two others were similar in form to the airtrack's poles in its *unstable* configuration.

- In Section 4, Part c, we examined the behavior of the system's poles as a function of the angular momentum of the two spacecraft spinning about the common origin. We found that as the nominal angular momentum increases, all the non-zero poles move toward infinity along the axes on which they lie. Eight of these poles behave in a similar manner when the spacecraft mass or separation distance increases.

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