16.61

LECTURE #3

- DYNAMICS "$F = ma$"
- VARIOUS EXAMPLES
- NUMERICAL INTEGRATION

NEWTON'S LAWS

① BODY CONTINUES IN ITS STATE OF MOTION OR REST UNLESS FORCED

② $\frac{d^I}{dt}(m\vec{v}) = \vec{F}$ — DIRECTIONS IMPORTANT

③ $\vec{F}_{12} = -\vec{F}_{21}$

MUST BE AN INERTIAL ACCELERATION.

- APPLY THESE LAWS TO A PARTICLE (m CONSTANT)

i) $\vec{F} = m\ddot{\vec{r}}^{I}$

ii) IF MANY FORCES ACT ON A PARTICLE, THEY CAN BE COMBINED VECTORIALLY

$$\vec{F} = \sum_{j=1}^{n} \vec{F}_j$$

(ASSUME FOR NOW THAT THESE ALL PASS THROUGH THE SAME POINT)

- **NOTE:** WHILE THE DERIVATIVES MUST BE EVALUATED WRT INERTIAL FRAME, WE CAN EXPRESS THE VECTORS $(\vec{F}, \ddot{\vec{r}}^{I})$ IN WHATEVER FRAME IS MOST CONVENIENT — BE CONSISTENT!!

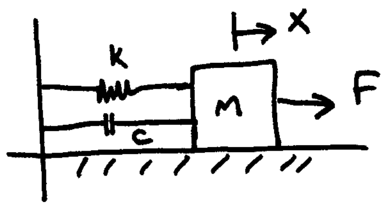
- <u>D' ALEMBERTS PRINCIPLE</u>

  - ALLOWS US TO CONVERT DYNAMICS TO STATICS

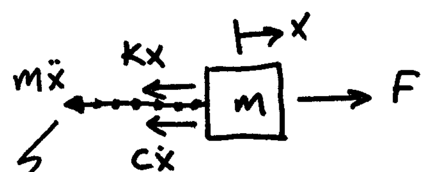
$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}^{I}$$

$$\Rightarrow \quad \vec{F} - \underbrace{m\ddot{\vec{r}}^{I}} = 0$$

- TREAT THIS AS A "FORCE" IN THE STATIC FORCE BALANCE
- MASS TIMES A REVERSED ACCELERATION



FORM FBD:



- D'ALEMBERT "FORCE"

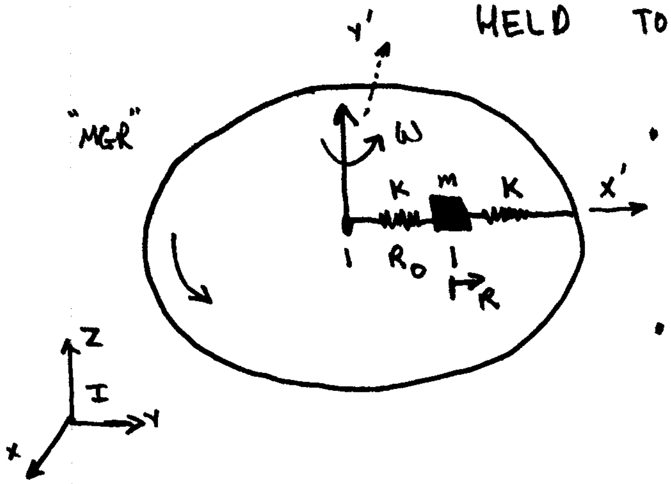
$\Rightarrow$ DO FORCE BALANCE:

STATICS $\rightarrow$ $\quad F - kx - c\dot{x} - m\ddot{x} = 0$

$(\Sigma F_j = 0)$

- SIMPLIFIES FORMULATION OF E.O.M.

① **EXAMPLE:** ROTATING DISK (MGR) WITH A MASS HELD TO CENTER BY A SPRING.



- ASSUME CONSTANT ANGULAR RATE SO THAT $\vec{\omega} = \omega \vec{K}$
- MASS m HELD BY 2 SPRINGS AT NOMINAL POSITION R_0 FROM CENTER – ADDITIONAL MOTION DENOTED BY R.

"M"

- DEFINE SECOND FRAME ATTACHED TO MGR WITH X-AXIS (X') IN DIRECTION OF THE SPRINGS.

$\Rightarrow$ LET $P_M = \begin{bmatrix} R_0 + R \\ 0 \\ 0 \end{bmatrix}$ DENOTE THE POSITION OF THE MASS.

– R_0 ~ NEUTRAL DISPLACEMENT OF SPRINGS.

- WILL ASSUME THAT MASS SLIDES IN A RADIAL SLOT WITH NO FRICTION.

- WITH MASS MOTION OF "R", THE TWO SPRINGS WILL EXERT A RESTORING FORCE OF $-2KR$ WHICH WILL ACT IN THE X' DIRECTION

$\Rightarrow$ IN THE SECOND FRAME. (CALL THIS "M")

- $\dot{P}^M_M = \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix}$; $\ddot{P}^M_M = \begin{bmatrix} \ddot{R} \\ 0 \\ 0 \end{bmatrix}$; $\omega^X_M = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- LET US FIND THE ABSOLUTE ACCELERATION OF THE MASS AND WRITE THAT IN THE SECOND FRAME AS WELL.

$$\ddot{\vec{p}}^{I} = \ddot{\vec{p}}^{M} + \cancel{\dot{\vec{\omega}}^{I} \times \vec{p}}^{\,0} + 2\vec{\omega} \times \dot{\vec{p}}^{M} + \vec{\omega} \times (\vec{\omega} \times \vec{p})$$

$\Rightarrow$ CHOOSE TO USE THE "M" FRAME, NOW USE MATRIX NOTATION

$$\ddot{p}^{I}_{M} = \ddot{p}^{M}_{M} + 2\omega^{\times}_{M} \dot{p}^{M}_{M} + \omega^{\times}_{M} \omega^{\times}_{M} p_{M}$$

$$= \begin{bmatrix} \ddot{R} \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 + R \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{R} - \omega^2 (R_0 + R) \\ 2\dot{R}\omega \\ 0 \end{bmatrix}$$

- KNOW THAT $F_M = m \ddot{p}^{I}_{M}$

$$\Rightarrow \quad -2KR = m(\ddot{R} - \omega^2 R) \qquad \Rightarrow \ddot{R} + \left(\frac{2K}{m} - \omega^2\right) R = R_0 \omega^2$$

AND $F_y = 2\omega\dot{R}$

- MASS MOTION – COMPARED TO MOTION OF MASS WITHOUT THE MGR ?

- SOLUTION OF $\ddot{R} + \Omega^2 R = R_0 \omega^2$ $\quad \Omega^2 > 0$

OF THE FORM

$$R(t) = \frac{R_0 \omega^2}{\Omega^2} + A_0 \cos \Omega t + A_1 \sin \Omega t$$

- A_0, A_1 RELATED TO THE INITIAL CONDITIONS $R(0)$ AND $\dot{R}(0)$

- SOLUTION IS SINUSOIDAL OSCILLATION
  - OSCILLATION FREQUENCY Ω
  - NOTE THAT SPIN ω TENDS TO REDUCE Ω

    SINCE $\Omega^2 = \frac{2K}{M} - \omega^2 > 0$

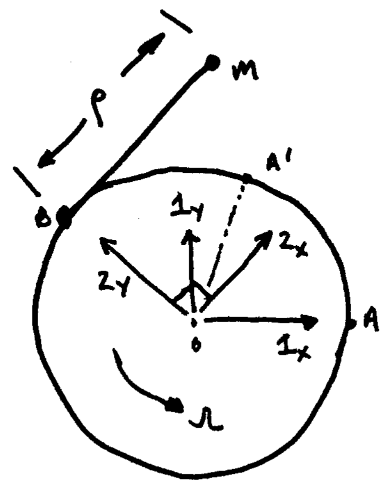
    $\omega_s = \sqrt{\frac{2K}{M}}$ ⇐ NATURAL FREQ OF OSCILLATION WITH NO SPIN.

- BUT NOTE THAT THERE IS A SPIN RATE ω_c FOR WHICH $\Omega^2 = \frac{2K}{M} - \omega_c^2 \equiv 0$
  - FOR $\omega > \omega_c$, $\Omega^2 < 0$

    ⇒ SOLUTION CHANGES TO

$$R(t) \sim \alpha e^{\gamma_1 t} + \beta e^{\gamma_2 t} \qquad \begin{array}{l} \gamma_1 < 0 \\ \gamma_2 > 0 \end{array}$$

2 **EXAMPLE: MORE ON FRAME SELECTION**



- MGR ROTATING AT RATE Ω
- MASSLESS STRING WRAPPED AROUND MGR, WITH A MASS m ATTACHED – MGR RADIUS a
- MASS INITIALLY LOCATED AT "A". IN TIME t, POINT A HAS ROTATED TO A', AND THE MASS HAS SWUNG OUT TO THE POSITION SHOWN

- WHAT IS THE ACCELERATION OF m $\Rightarrow$ DIFFERENTIAL EQUATION FOR ρ $\Rightarrow$ VERY SIMILAR TO SPACECRAFT DEVICE USED TO SLOW SPIN.

• FRAME SELECTION. – NOTE THAT WE CAN ASSUME THAT THE STRING IS TANGENT AT POINT B.
 $\Rightarrow$ NEED TO TRACK POINT B AND WANT AN EASY WAY TO SPECIFY LOCATION OF m.
 $\Rightarrow$ CHOOSE SECOND FRAME SO THAT Y-AXIS PASSES THROUGH B FROM O

$$\Rightarrow \rho_2 = \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix}$$ POSITION OF m WRT O IN FRAME 2

• ASSUME ONLY FORCE ACTING ON THE MASS IS THE TENSION IN THE ROPE $F_2 = \begin{bmatrix} -T \\ 0 \\ 0 \end{bmatrix} \leftarrow 2x$

- SO FAR OK, BUT <u>WHAT IS THE ANGULAR RATE</u> BETWEEN THESE TWO FRAMES?

  - LET θ BE THE ANGLE FROM 1_x TO 2_y
  - ANGLE FROM A TO A' $\rightarrow \int \Omega \, dt$
  - ANGLE FROM A' TO B $\rightarrow \dfrac{\rho}{a}$

  WHY? ρ = LENGTH OF STRING UNWOUND $\rightarrow$ ARC LENGTH FROM A'

$$\Rightarrow \quad \theta = \int \Omega \, dt + \frac{\rho}{a} \qquad \text{ABOUT Z-AXIS}$$

$$\therefore \quad \dot\theta = \Omega + \frac{\dot\rho}{a} \qquad \Rightarrow \quad \vec{\omega} = \dot\theta \, \vec{K}$$

$$\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix} \; ; \; \omega_2^{\times} = \begin{bmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ddot\theta = \cancel{\dot\Omega}^{\,0} + \frac{\ddot\rho}{a}$$

- $$\ddot{\vec{\rho}}^{\,I} = \ddot{\vec{\rho}}^{\,R} + \dot{\vec{\omega}}^{\,I} \times \vec{\rho} + 2\vec{\omega} \times \dot{\vec{\rho}}^{\,R} + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})$$

  CHOOSE TO WRITE THIS IN FRAME 2 $\rightarrow$ USE MATRIX FORM.

$$\ddot\rho_2^{\,I} = \begin{bmatrix} \ddot\rho \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\ddot\rho/a & 0 \\ \ddot\rho/a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\rho \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot\theta^2 & 0 & 0 \\ 0 & -\dot\theta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot\rho - \ddot\rho - \rho\dot\theta^2 \\ \rho\ddot\rho/a + 2\dot\rho\dot\theta - a\dot\theta^2 \\ 0 \end{bmatrix} = \frac{F_2}{m} = \frac{1}{m} \begin{bmatrix} -T \\ 0 \\ 0 \end{bmatrix}$$

- SO, FOR ρ, WE MUST SOLVE

$$\frac{\rho\ddot{\rho}}{a} + 2\dot{\rho}\left(\Omega + \frac{\rho}{a}\right) - a\left(\Omega + \frac{\dot{\rho}}{a}\right)^2 = 0$$

$$\Rightarrow \rho\ddot{\rho} + \cancel{2a\dot{\rho}\Omega} + 2\dot{\rho}^2 - a^2\Omega^2 - a^2\left(\cancel{\frac{2\Omega}{a}\dot{\rho}}\right) - \dot{\rho}^2 = 0$$

$$\Rightarrow \rho\ddot{\rho} + \dot{\rho} = a^2\Omega^2$$

$$\Rightarrow \frac{d}{dt}(\rho\dot{\rho}) = a^2\Omega^2$$

$\therefore \quad \rho\dot{\rho} = a^2\Omega^2 t + C_1$ AT $t=0,\ \rho=0 \Rightarrow C_1 = 0$

$\Rightarrow \quad \rho^2 = a^2\Omega^2 t^2 + C_2$ (INTEGRATE BOTH SIDES)

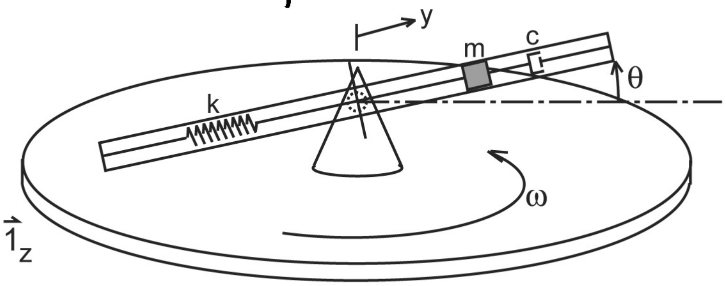
$C_2 = 0$ ALSO

$$\therefore \quad \rho = \pm a\Omega t$$

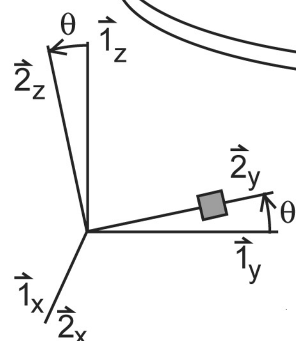
- HOW WOULD YOU FIND T?

- SOLUTION SIMPLIFIED BY APPROPRIATE SELECTION OF FRAME 2.

(3) **EXAMPLE:** ROTATING PLATFORM CARRYING A TUBE WITH A MASS IN IT THAT IS HELD BY A SPRING. THE PLATFORM IS ROTATING AT A GIVEN RATE Ω, AND THE TUBE CAN OS



- ATTACH FRAME 1 TO THE MGR WITH THE Y-AXIS ALIGNED WITH THE TUBE



- SELECT A SECOND FRAME THAT IS ATTACHED TO THE TUBE. GET FROM FRAME 1 TO 2 WITH A ROTATION OF θ ABOUT $\vec{1}_x\ (\vec{2}_x)$

- ASSUME THAT THE NEUTRAL POSITION FOR THE SPRING IS $y_0 = 0 \Rightarrow$ MASS LOCATION WRT O IS

$$P_2 = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

- TOTAL ANGULAR VELOCITY OF FRAME 2 WRT INERTIAL IS $\vec{\omega} = \Omega \vec{1}_z + \dot{\theta}\, \vec{2}_x$

- PLAN TO COMPUTE ACCELERATIONS AND REPRESENT THEM IN THE 2-FRAME (ROTATION BY θ)

$$W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \Omega \sin\theta \\ \Omega \cos\theta \end{bmatrix}$$

$$\begin{matrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{matrix}$$

- ACCELERATION IN 2-FRAME

$$\ddot{p}_2^I = \ddot{p}_2^2 + \dot{\omega}_2^\times p_2 + 2\omega_2^\times \dot{p}_2^2 + \omega_2^\times \omega_2^\times p_2$$

$$\dot{\omega}_2 = \begin{bmatrix} \ddot{\theta} \\ \Omega \cos\theta \, \dot{\theta} \\ -\Omega \sin\theta \, \dot{\theta} \end{bmatrix} ; \quad \dot{\omega}_2^\times = \begin{bmatrix} 0 & \Omega \sin\theta \, \dot{\theta} & \Omega \cos\theta \, \dot{\theta} \\ -\Omega \sin\theta \, \dot{\theta} & 0 & -\ddot{\theta} \\ -\Omega \cos\theta \, \dot{\theta} & \ddot{\theta} & 0 \end{bmatrix}$$

$$\ddot{p}_2^I = \begin{bmatrix} 0 \\ \ddot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} y \Omega \sin\theta \, \dot{\theta} \\ 0 \\ y \ddot{\theta} \end{bmatrix} + 2 \begin{bmatrix} -\Omega \cos\theta \, \dot{y} \\ 0 \\ \dot{y} \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -\Omega \cos\theta & \Omega \sin\theta \\ \Omega \cos\theta & 0 & -\dot{\theta} \\ -\Omega \sin\theta & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} -y \Omega \cos\theta \\ 0 \\ y \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} y \Omega \sin\theta \, \dot{\theta} - 2 \Omega \cos\theta \, \dot{y} + y \Omega \sin\theta \, \dot{\theta} \\ \ddot{y} - y \Omega^2 \cos^2\theta - y \dot{\theta}^2 \\ y \ddot{\theta} + 2 \dot{y} \dot{\theta} + y \Omega^2 \sin\theta \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2\Omega \left(y \dot{\theta} \sin\theta - \dot{y} \cos\theta \right) \\ \ddot{y} - y \left(\dot{\theta}^2 + \Omega^2 \cos^2\theta \right) \\ y \ddot{\theta} + 2 \dot{y} \dot{\theta} + y \Omega^2 \sin\theta \cos\theta \end{bmatrix}$$

- PRETTY UGLY, NONLINEAR DYNAMICS

- FORCES
  - SPRING / DASHPOT – ACT ALONG $\vec{2}_Y$

$$F_{SD} = -(ky + c\dot{y})\,\vec{2}_Y$$

  - GRAVITY – ACTS ALONG $\vec{1}_Z$

$$F_G = -mg\,\vec{1}_Z$$

  - NORMAL FORCE FROM TUBE – ACTS ALONG $\vec{2}_Z$

    BUT NOT IMPORTANT HERE

$$\vec{F} = -(ky + c\dot{y})\,\vec{2}_Y - mg\,\vec{1}_Z$$

$$F_2 = \begin{bmatrix} 0 \\ -(ky+c\dot{y}) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} = \begin{bmatrix} 0 \\ -(ky+c\dot{y}+mg\,S\theta) \\ -mg\,C\theta \end{bmatrix}$$

- EQUATE $\quad F_2 = m\ddot{P}_2^{I}$

$\Rightarrow$ DIFFERENTIAL EQUATION FOR y IS:

$$m\left[\ddot{y} - y(\dot{\theta}^2 + \Omega^2\cos^2\theta)\right] = -ky - c\dot{y} - mg\sin\theta$$

$$m\ddot{y} + c\dot{y} + (k - m\dot{\theta}^2 - m\Omega^2\cos^2\theta) = -mg\sin\theta$$

$$m\left[y\ddot{\theta} + 2\dot{y}\dot{\theta} + y\Omega^2\sin\theta\cos\theta\right] = -mg\cos\theta$$

$\Rightarrow$ COULD SOLVE FOR BOTH $y(t)$, $\theta(t)$.

BUT NONLINEAR, SO NEED A NUMERICAL TECHNIQUE.

- WHAT IF WE HAD DECIDED TO WORK IN FRAME 1 INSTEAD?

  - EXPRESSION FOR ω_1 SIMPLIFIED SINCE NO ROTATIONS ARE REQUIRED.

$$\vec{\omega} = \Omega \vec{1}_z + \dot{\theta} \vec{2}_x \qquad \text{BUT} \qquad \vec{2}_x \equiv \vec{1}_x$$

$$\Rightarrow \quad \omega_1 = \begin{bmatrix} \dot{\theta} \\ 0 \\ \Omega \end{bmatrix}$$

  - EXPRESSION FOR $\vec{P}$ MORE COMPLEX IN FRAME 1

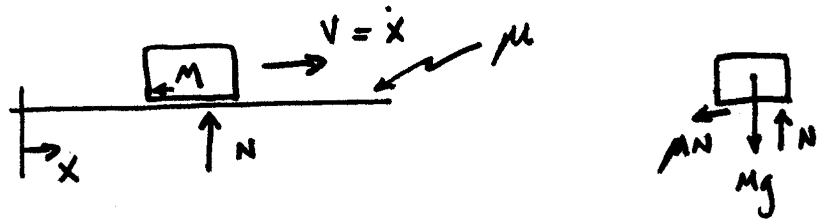
$$P_1 = \begin{bmatrix} 0 \\ y c\theta \\ y s\theta \end{bmatrix} \; ; \; \dot{P}_1^1 = \begin{bmatrix} 0 \\ -y s\theta \dot{\theta} + \dot{y} c\theta \\ y c\theta \dot{\theta} + \dot{y} s\theta \end{bmatrix}$$

$$\ddot{P}_1^1 = \begin{bmatrix} 0 \\ -\dot{y} s\theta \dot{\theta} - y s\theta \ddot{\theta} - y c\theta \dot{\theta}^2 + \ddot{y} c\theta - \dot{y} s\theta \dot{\theta} \\ \dot{y} c\theta \dot{\theta} + y c\theta \ddot{\theta} - y s\theta \dot{\theta}^2 + \ddot{y} s\theta + \dot{y} c\theta \dot{\theta} \end{bmatrix}$$

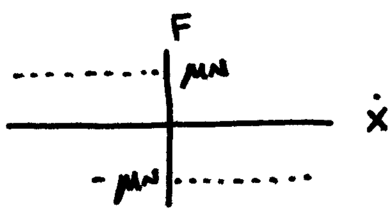
- COULD KEEP GOING, BUT CAN CLEARLY SEE THAT THIS IS MESSY WHEN COMPARED TO USING FRAME 2

  $\Rightarrow$ FRAME SELECTION CRUCIAL PART OF MAKING PROBLEM TRACTABLE.

COULOMB FRICTION



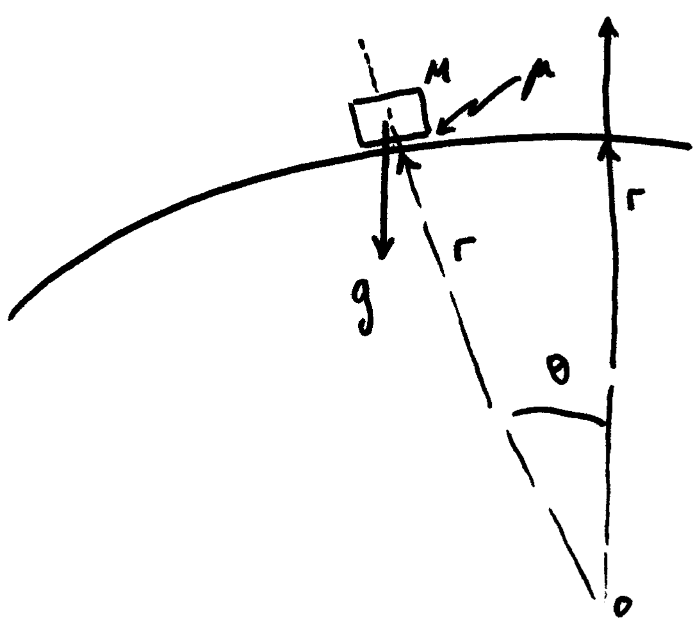
FRICTION FORCE $\quad F = -\,\mathrm{SGN}(v)\,\mu N$



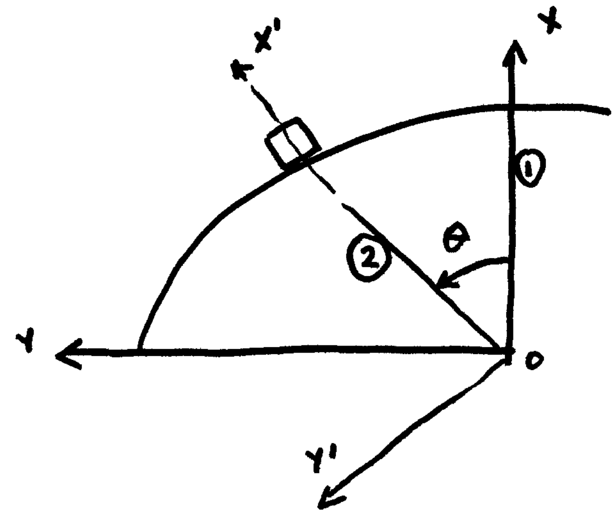
FRICTION OPPOSES MOTION

EXAMPLE: FIND E.O.M. FOR MASS SLIDING ON SPHERE - RADIUS R, FRICTION μN

⇒ WHEN DOES IT LEAVE THE SURFACE?



WHAT FRAMES MAKE SENSE HERE?



$$\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix} \quad \dot\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta \end{bmatrix}$$

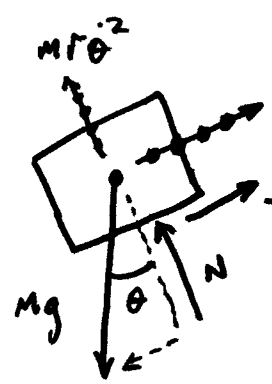
$\vec{r}$: POSITION OF MASS WRT ORIGION

$$\ddot r_2^{I} = \cancel{\ddot r_2^{R}}^{\,0} + \dot\omega_2^{\times} r_2 + \cancel{\omega_2^{\times} \dot r_2} + \omega_2^{\times}\omega_2^{\times} r_2$$

ON SPHERE

$$= \begin{bmatrix} -r\dot\theta^2 \\ r\ddot\theta \\ 0 \end{bmatrix} \longrightarrow \text{D'ALEMBERT FORCES IN ②} = -m \begin{bmatrix} -r\dot\theta^2 \\ r\ddot\theta \\ 0 \end{bmatrix} \begin{array}{l} x' \\ y' \\ z' \end{array}$$

FORCES?



$\longrightarrow \mu N\,SGN(r\dot\theta)$

RELATIVE VELOCITY OF MASS

$$F_2 = \begin{bmatrix} N + mr\dot\theta^2 - Mg\cos\theta \\ Mg\sin\theta - mr\ddot\theta - \mu N\,SGN(r\dot\theta) \\ 0 \end{bmatrix}$$

E.O.M. $\Rightarrow$ $F_2 = 0$

FOR DEPARTURE ANGLE, SOLVE FOR θ. CHECK $N \geq 0$?