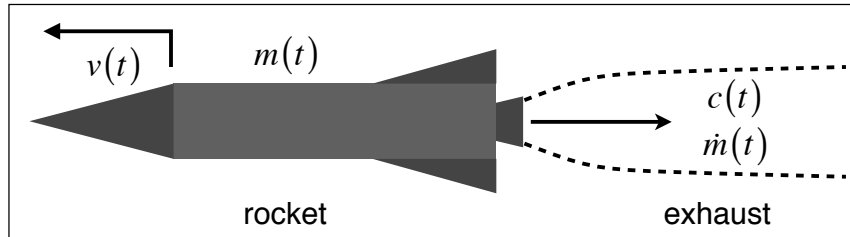


## Session 2: Fundamentals and Definitions

The only practical way to accelerate something in free space is by reaction. The idea is the same as in air breathing propulsion (to push something backwards) but in rockets the “something” must be inside and is lost. Even though newtonian mechanics were very well established by the late 19th century, many scientists and engineers had difficulty conceiving that reaction propulsion in the vacuum of space would be feasible at all. Of course, now we know that this is possible and since 1957 with the launch of Sputnik, practical. Let us start by analyzing the rocket as a variable-mass system operating in vacuum. Consider a system composed by a rocket and its exhaust enclosed in a controlled volume as depicted below. The volume “shrink-wraps” the exhaust such that no mass at all leaves the system.



The momentum enclosed in this volume can be written as,

$$P_m(t) = m(t)v(t) + \int_0^t \dot{m}(t') [v(t') - c(t')] dt' \quad (1)$$

We also assume a gravity-free environment with no external forces acting on the system. Therefore, the enclosed momentum must be constant, i.e.,

$$\frac{dP_m(t)}{dt} = 0 \quad (2)$$

The derivation then proceeds on the right hand side of Eq. (1),

$$m \frac{dv}{dt} + v \frac{dm}{dt} + \dot{m}(v - c) = 0 \quad (3)$$

The mass flow rate of the exhaust is equal in magnitude (opposite in direction) to the rate of mass lost by the rocket, or  $\dot{m} = -dm/dt$ . Using this and performing a few cancellations in Eq. (3) we get,

$$m \frac{dv}{dt} = -c \frac{dm}{dt} \quad \text{or} \quad \frac{dv}{c} = -\frac{dm}{m} \quad (4)$$

The right hand side version integrates directly into the well-known form of the rocket equation for a vehicle of initial mass  $m_0$ :

$$m_f = m_0 \exp\left(-\frac{\Delta v}{c}\right) \quad (5)$$

The propellant use in a particular mission for a given velocity change  $\Delta v = v_f - v_0$  and exhaust velocity  $c$ , is simply  $m_p = m_0 - m_f$ , or,

$$m_p = m_0 \left[ 1 - \exp \left( -\frac{\Delta v}{c} \right) \right] \quad (6)$$

From a complexity and economic perspective, low  $m_p/m_0$  is obviously a positive feature. In rocket propulsion, this basically means low  $\Delta v/c$ , which for the most interesting applications is achieved not by decreasing the value of  $\Delta v$ , but by increasing the magnitude of the relative exhaust velocity  $c$ .

It is important at this point to clarify that except for impulsive maneuvers and continuous thrust travel far away from the influence of the sun and planets,  $\Delta v$  in Eqs. (5-6) is not in general equal to the actual change in velocity of the spacecraft. For example, take a thruster firing continuously at LEO to overcome high-altitude drag. In this case, there is no change in spacecraft velocity, but still a  $\Delta v$  could be associated to the mission as defined by the propellant mass consumption through Eq. (6).

The left hand side version of Eq. (4) is quite revealing. The  $m(dv/dt)$  term represents the instantaneous mass multiplying the instantaneous acceleration of the rocket. This is the force, or thrust, acting in the direction of motion,

$$F = -c \frac{dm}{dt} \quad \text{or} \quad F = \dot{m}c \quad (7)$$

Now, assume we apply a force to a vehicle for a given time  $t$ . We calculate the total impulse (or net change of momentum) imparted to the vehicle,

$$I = \int_0^t F dt' = \int_0^t \dot{m}c dt' \quad (8)$$

The objective of most propulsion systems is to impart the largest possible impulse to its carrying vehicle. From Eq. (8) it is trivial to see that a very large impulse could be attained if the force is applied over a long time. Impulse then depends on how big the propellant tank is, shedding little light into how effectively the propulsion system delivers that impulse to fulfill mission requirements.

To remove the dependence on tank size and more importantly, to focus on thruster performance, we compute instead the specific impulse, or impulse per unit weight of propellant,

$$I_{sp} = \frac{\int_0^t F dt'}{W_p} = \frac{\int_0^t \dot{m}c dt'}{g \int_0^t \dot{m} dt'} \quad (9)$$

For constant exhaust velocity, the specific impulse becomes,

$$I_{sp} = \frac{c}{g} \quad [\text{seconds in mks}] \quad (10)$$

Having the specific impulse in units of time is more of a convenience than anything else for space thrusters. We gain little insight in looking for a physical interpretation of the specific

impulse in terms of its value in units of time. However, this value is particularly relevant for launchers. The specific impulse represents the time a particular engine can operate when its thrust equals the initial weight of the propellant used by it. Given that launcher engines need to provide a thrust to weight ratio of one or more, then  $I_{sp}$  gives an idea of the time the rocket will have before running out of propellant, and therefore whether or not it will achieve its mission. In some calculations, carrying the  $g$  is cumbersome and we therefore will refer in many instances to  $c$  as the specific impulse, with the understanding that the set of units need to be consistent with the particular situation under consideration.

Another interesting feature surfaces if we calculate the rate of change of the momentum of the rocket. From Eq. (1),

$$\frac{d(mv)}{dt} = \dot{m}(c - v) \quad (11)$$

We observe that the right hand side of this equation can be positive, negative or zero. Even though the system is effectively accelerated, propellant is constantly used and the rocket will lose momentum when its velocity increases beyond  $c$ .

In a similar way, we calculate the rate of change of kinetic energy of the rocket,

$$K_R = \frac{1}{2}mv^2 \quad \text{and} \quad \frac{dK_R}{dt} = mv\frac{dv}{dt} + \frac{1}{2}v^2\frac{dm}{dt} = \dot{m}v\left(c - \frac{v}{2}\right) \quad (12)$$

where the final result is obtained after using Eq. (4). In this case, once more because of mass ejection, the rocket will start losing kinetic energy when its velocity increases beyond  $2c$ . It is also easy to see that the maximum kinetic power delivered to the rocket occurs when  $v = c$ . In this case, from the point of view of an inertial observer, the exhaust stands still as it is ejected from the rocket, and therefore all energy goes into vehicle acceleration. However, notice that as stated, Eq. (12) ignores the motion of the exhaust, but still the math retains information of its presence. This should not be surprising, since in a well-posed problem like this, physics guarantees self-consistency.

We now turn our attention to the kinetic energy of the rocket-exhaust system,

$$K_T = \frac{1}{2}mv^2 + \int_0^t \frac{1}{2}\dot{m}(v - c)^2 dt' \quad (13)$$

We then calculate the net rate of change of the total kinetic energy,

$$\frac{dK_T}{dt} = \frac{dK_R}{dt} + \frac{1}{2}\dot{m}(v - c)^2 \quad \text{then} \quad \frac{dK_T}{dt} = \frac{1}{2}\dot{m}c^2 \quad (14)$$

This is a very important result. It fundamentally means that the net rate of change of kinetic energy of the rocket-exhaust system does not depend on the instantaneous velocity, but only on the mass flow rate and the relative exhaust velocity. In physical terms, this is the kinetic power delivered to all the mass in the system. Some of that power goes into accelerating the rocket, the rest is used to accelerate the exhaust. In any event, this power must come from an energy source. The efficiency of the propulsion system is defined as the fraction of the total source power which is transformed into kinetic (or jet) power,

$$\eta = \frac{\frac{1}{2}\dot{m}c^2}{P} \quad (15)$$

If we try to define useful propulsive work rate (like in aircraft engines) as  $Fv = \dot{m}cv$ , then we find that the propulsive efficiency,

$$\eta_{prop} = \frac{\dot{m}cv}{P} = \eta \frac{\dot{m}cv}{\frac{1}{2}\dot{m}c^2} = 2\eta \frac{v}{c} \quad (16)$$

is arbitrarily high (if  $v > c/2$  then  $\eta_{prop} > 1$  for  $\eta = 1$ ). Therefore, the propulsive efficiency is not useful to describe the performance of rockets.

In these lectures we will focus mostly on rockets operating in the vacuum of space with very rarified exhausts, which from a thermodynamics perspective expand ideally to zero pressure (typical conditions for electric propulsion engines). Nevertheless, many rockets (mostly chemical thrusters) operate inside an atmosphere with non-ideal expansions, and it is useful at this point to remind ourselves about the difference between these two operating modes. Eq. (7) provides an unambiguous definition of thrust in which the specific impulse and the ideal exhaust velocity  $c$  are directly linked. However, in general, a rocket's thrust can be decomposed in two terms,

$$F = \dot{m}u_e + (p_e - p_a)A_e \quad (17)$$

Assuming that propellant particles accelerate from zero relative velocity, the exhaust velocity  $u_e$  is found in terms of the added energy per unit mass  $\varepsilon = P/\dot{m}$  and the thermodynamic efficiency  $\eta_t$ ,

$$u_e = \sqrt{2\eta_t\varepsilon} \quad (18)$$

In a chemical rocket, the energy per unit mass includes contributions of the internal energy and pressure work, which is the enthalpy per unit mass  $h = c_p T$ . Eq. (18) then becomes,

$$u_e = \sqrt{2\eta_t c_p T_0 \left[ 1 - \left( \frac{p_e}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right]} \quad (19)$$

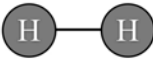
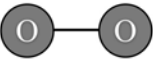
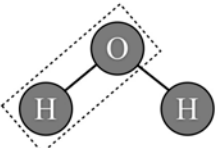
The exhaust velocity and the specific impulse  $c = F/\dot{m}$  are not the same. They might be close in many circumstances, but in general they differ by the pressure term on the right in Eq. (17). When operating in vacuum ( $p_a = 0$ ), we have direct jet thrust  $\dot{m}u_e$  and pressure thrust  $p_e A_e$ . Increasing the pressure thrust at the expense of poor expansion is not recommended, since jet thrust decreases more rapidly than the corresponding increase of pressure thrust as  $p_e$  increases.

If we neglect pressure thrust, the specific impulse and exhaust velocity are the same and both can be computed through the use of Eq. (18). This is beneficial in understanding the fundamental differences between chemical and electric propulsion engines.

Let us begin with chemical rockets. Take for instance the stoichiometric reaction of hydrogen and oxygen, one of the most energetic reactions with practical use in chemical thrusters,



The available energy for propulsion is effectively stored in the bonds that form the molecules of reactants and products:

H <sub>2</sub>		436 kJ/mol
O <sub>2</sub>		498 kJ/mol
H <sub>2</sub> O		OH bond: 428 kJ/mol OH-H bond: 498.7 kJ/mol

The energy per unit mass of water molecules (molecular weight = 18 g/mol) in Eq. (18) is the difference between the bond energy of products and reactants,

$$\varepsilon = \frac{\varepsilon_p^{bond} - \varepsilon_r^{bond}}{m_p} = \frac{[(428 + 498.7) - (436 + \frac{1}{2}498)] \times 10^3}{0.018} = 1.34 \times 10^7 \text{ J/kg} \quad (21)$$

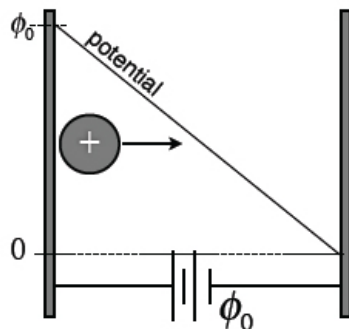
Assuming an efficiency of 90%, the specific impulse becomes,

$$c = \sqrt{2\eta\varepsilon} = 4917 \text{ m/s} \quad \text{or} \quad I_{sp} = \frac{c}{g} = 502 \text{ s} \quad (22)$$

This is an important result. The maximum specific impulse for a chemical rocket is about 500 seconds. This is close to what is achieved with efficient expander (RL-10, 462 sec) and staged combustion (SSME, 453 sec) cycles. What this means is that chemical rockets have no practical way of improving their  $\Delta v$  capability for a given amount of propellant used. This does not mean that chemical rockets are perfect, since there is still significant work to be done towards making chemical engines cheaper and safer. We used the word “practical” before, since there are a few extraordinarily energetic chemical reactions that could, in principle, be used in rockets. For instance, direct combustion of atomic hydrogen would yield 436 kJ/mol, and given the low molecular weight of the H<sub>2</sub> product, we get an  $I_{sp}$  of 2000 sec. This “atomic” rocket would revolutionize space travel. Unfortunately, the technology to store and deliver atomic hydrogen does not currently exist.

Let us now assume the water molecules that comprise the exhaust are again accelerated to the same velocity as in Eq. (22), but using electrical means. For this, we concentrate on what happens to a single water molecule which we take as positively charged after removing

one electron from one of its atoms. We place such charged molecule with zero velocity at the left electrode shown in the figure below and then apply an electric potential (voltage) with respect to a second downstream electrode (considered here at zero potential).



Ignoring space charge effects, we have a constant electric field pushing the ion to the right. Through a simple energy balance, we find an expression for the velocity of ions arriving at the second electrode. If such electrode were transparent to ions, we would then have a thruster with specific impulse,

$$\cancel{\frac{1}{2}m_i u_0^2} + e\phi_0 = \frac{1}{2}m_i c^2 + \cancel{e\phi} \quad \rightarrow \quad c = \sqrt{2\frac{e}{m_i}\phi_0} \quad (23)$$

Substituting the value of  $c$  from Eq. (22) and using the charge and mass of the water ion, we find an electric potential  $\phi_0 = 2.25$  V. The message in here is very clear: it is possible to obtain the same specific impulse as the most advanced chemical rocket by applying a very modest potential difference to an ion cloud. The promise of electric propulsion is to eliminate the specific impulse limitation of chemical rockets by applying electric potentials which are in effect much larger than a few volts (at least for the ion engine variety exemplified here).

Large specific impulse translates into propellant savings for any type of mission. However, from Eqs. (14-15), high specific impulse implies large power needs. The mass of the power generation and processing units grows roughly linearly with required power, which increases as the square of the specific impulse for constant mass flow rate, or linearly for constant thrust,

$$P = \frac{\dot{m}c^2}{2\eta} = \frac{Fc}{2\eta} \quad (24)$$

For example, take a small monopropellant (chemical) rocket delivering 10 N of thrust with a specific impulse of 200 seconds and efficiency of 90%. From Eq. (24) we find the (chemical) power required by this thruster to be 11 kW. If the same amount of power were required by an electric thruster, a separate source would be needed, like solar arrays plus a power processing unit (PPU). More than 30 m<sup>2</sup> of solar panel active area would be required to generate such power (assuming 25% efficient cells at 1 au). This is the main difference between chemical and electric rockets. In addition to propellant, electric rockets need to carry a power source. Nevertheless, the *fuel* efficiency of electric rockets make them very attractive for a number of missions and savings in propellant typically offset the increase in mass brought by the power source and additional electronics.

Finally, consider the specific power (power to spacecraft mass),

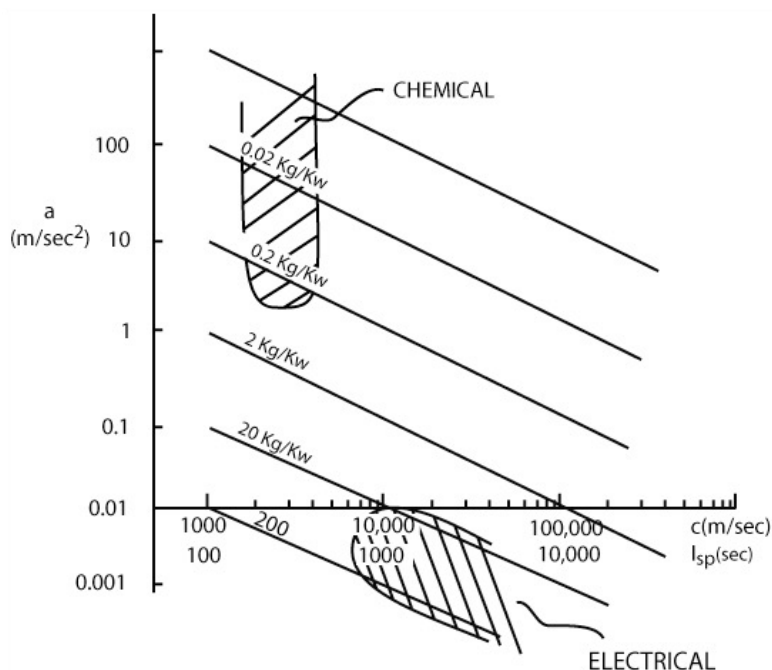
$$\frac{P}{m} = \frac{P}{m_{ps}} \frac{m_{ps}}{m} = \frac{F}{m} \frac{c}{2\eta} = a \frac{c}{2\eta} \quad (25)$$

For a fixed specific power, it is clear that high specific impulse means low acceleration. This is the main tradeoff with electric thrusters: savings in propellant are attractive, but due to power limitations the time to achieve a given mission will be long due to reduced accelerations.

To give a rough idea of the applicability domains for electric vs chemical thrusters, we plot acceleration against specific impulse for several values of specific mass (per unit power) of the propulsion system, defined as,

$$\alpha = \frac{m_{ps}}{P} \quad (26)$$

In this definition, the mass includes the thruster and PPU. For the plot we take unit efficiency and  $m \approx m_{ps}$ , which effectively means the maximum acceleration possible.



A fundamental part in the development of electric propulsion technology includes the reduction of  $\alpha$ , the specific mass of the propulsion and power system. In the plot above we marked the region of electrical thrusters with  $\alpha$  between 20 (nuclear-electric) and 200 kg/kW (solar-electric). With the advent of advanced electronics and multi-spectral high efficiency solar cells, it should be possible to reach values as low as 80-150 kg/kW. Predicting the low end for nuclear power is more difficult given the comparably low experience base of reactors in space, but in principle, it should be possible to reach values in the single digits of kg/kW, especially when using lightweight structures and efficient radiators that share cooling of the reactor with the thrusters.

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