16.522 Space Propulsion
Problem Set 2

The differential equations of motion derived in class can be solved numerically for any kind of accelerating forces acting on a vehicle. If we restrict ourselves to planar motion, we have, in polar coordinates:

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = a_r \qquad \text{and} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_\theta$$

where a_r and a_θ are the radial and angular accelerations so that $a_c^2 = a_r^2 + a_\theta^2$.

Your assignment consists on designing a software tool that integrates numerically a non-dimensional version of these equations. Do not use anything less accurate than a *Runge-Kutta* algorithm.

(a) Derive expressions for a_r and a_θ as a function of γ, the angle between the orbital velocity and thrust vectors.

(b) Write down the equations of motion in non-dimensional form by defining,

$$\hat{r} = \frac{r}{a_0} \qquad \hat{t} = \frac{t}{\sqrt{a_0^3/\mu}} \qquad \Delta\hat{v} = \frac{\Delta v}{\sqrt{\mu/a_0}} \qquad \text{and} \qquad \hat{a}_{r,\theta} = \frac{a_{r,\theta}}{\mu/a_0^2} \qquad (\varepsilon = \hat{a}_\theta)$$

where a_0 is the semi-major axis of the baseline (or initial) orbit.

(c) Assume angular accelerations only and run your code for several values of $\varepsilon = \hat{a}_\theta$ until reaching escape conditions.

- Reproduce the results in the table shown in page 4 of the notes (Lecture 5).
- Calculate the same parameters of the table in the case of tangential thrust and comment on the differences.
- Include a plot of the spiral trajectory with the highest ε and mark the escape point on your plot. Is the orbit nearly circular at that point?
- Include a plot of the orbital energy as a function of $\Delta\hat{v}$ for the same case.
- Compare against the impulsive $\Delta\hat{v}_{\text{escape}}$.
- Compare against the analytical expressions for r (spiral) and Δv_{esc} derived in class.

(d) As we discussed in class it is possible to write differential equations for the integrals of motion for low thrust maneuvers. The general method is relatively straightforward: start from the regular identities where the integrals of motion appear and take time derivatives treating these integrals of motion as variables and the orbital radius as constant. The derivative of the velocity vector is treated as the acceleration due to thrust. Use this procedure and,

Starting from	Prove that
$\mu\left(\frac{2}{r}-\frac{1}{a}\right) = v^2 = \vec{v}\cdot\vec{v}$	$\frac{da}{dt} = \frac{2a^2}{\mu}(\vec{v}\cdot\vec{a}_c)$
$\vec{h} = \vec{r}\times\vec{v}$	$\frac{dh}{dt} = \frac{1}{h}\left[r^2(\vec{v}\cdot\vec{a}_c) - (\vec{r}\cdot\vec{v})(\vec{r}\cdot\vec{a}_c)\right]$
$p = a(1-e^2)$	$\frac{de}{dt} = \frac{1}{\mu a e}\left[(\vec{r}\cdot\vec{v})(\vec{r}\cdot\vec{a}_c) + (pa - r^2)(\vec{v}\cdot\vec{a}_c)\right]$

Notes: These expressions are much easier to prove using tensorial notation. This method is somewhat unorthodox, however, it can be proved that the results are the same as when following the more rigorous variation of parameters approach.

(e) As an example, consider that low thrust is applied to a vehicle in an initially elliptical orbit (say, $e_0 = 0.5$). It can be proven that firing always perpendicular to the apsidal line will circularize the orbit without changing the semi-major axis. As a side note, this maneuver is patented by a satellite manufacturer and is used to circularize geosynchronous orbits. Use your numerical tool to prove this.

- Plot the firing angle γ as a function of time for one orbital period.
- Plot the non-dimensional version of the quantities derived in (d).
- Show the resulting trajectory in a polar plot.
- Plot the eccentricity as a function of $\Delta\hat{v}$.