16.512, Rocket Propulsion
Prof. Manuel Martinez-Sanchez
Lecture 34: Performance to GEO

ΔV Calculations for Launch to Geostationary Orbit (GEO)

Idealized Direct GTO Injection
(GTO = Geosynchronous Transfer Orbit)

Assumptions:

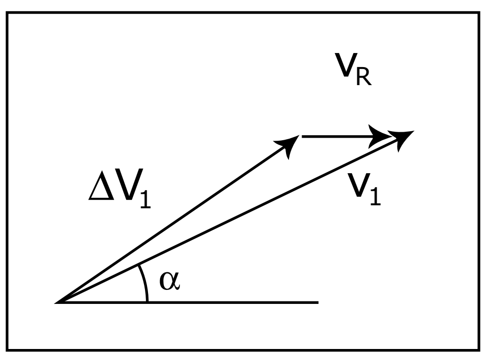
- Ignore drag and "gravity" losses
- Assume impulsive burns (instantaneous impulse delivery)
- Assume all elevations $\alpha>0$ at launch are acceptable

Launch is from a latitude L, directed due East for maximum use of Earth's rotation. The Eastward added velocity due to rotation is then

$$v_R = \Omega_E R_E \cos L = 463 \cos L \quad (m/s) \tag{1}$$

If the launch elevation is α, and the desired velocity after the first burn is V_1, the rocket must supply a velocity increment

$$\Delta V_1 = \sqrt{V_1^2 + v_R^2 - 2 V_1 v_R \cos\alpha} \tag{2}$$



The trajectory will then lie in a plane LOI through the Earth's center which contains the local E-W line. In order to be able to perform the plane change to the equatorial plane at GEO, we select the elevation α such as to place the apogee of the transfer orbit (GTO) at the GEO radius $R_{GEO} = \left(\mu \frac{T^2}{4\pi^2}\right)^{1/3} = 42,200$ km

(T = 24 hr, $\mu = 3.986\times10^{14}$ m^3/s^2)

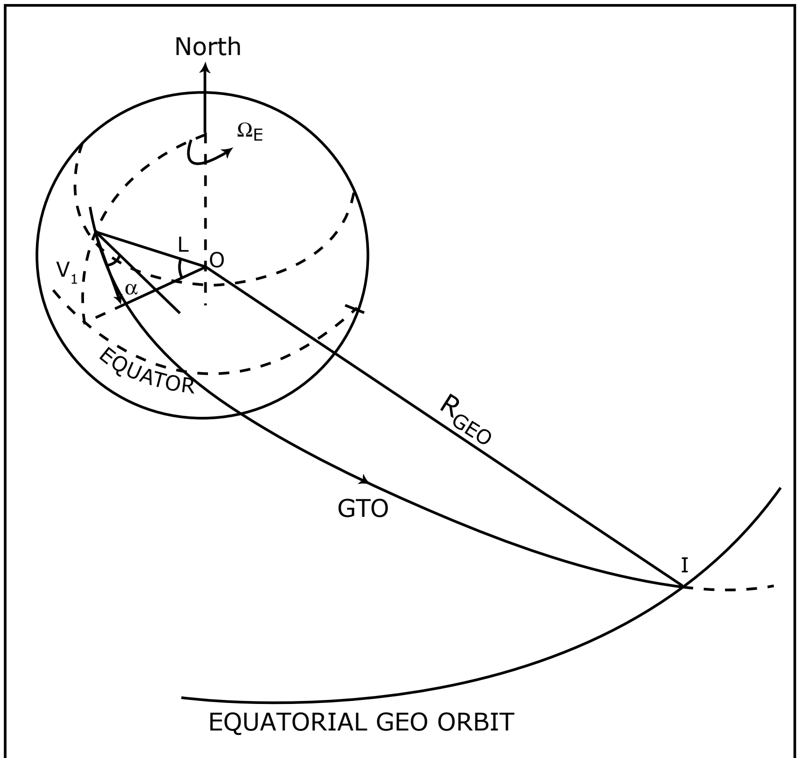


Fig. 1

Since OL is perpendicular to OI, the view in the plane of the orbit is:

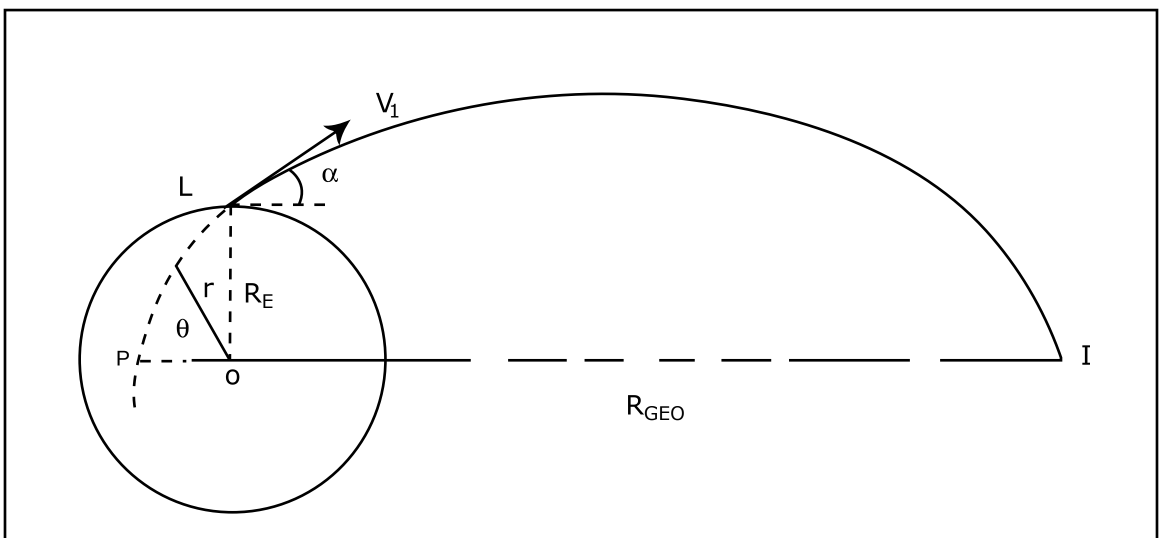


Fig. 2

The polar equation of the trajectory is $r = \frac{p}{1 + e\,\cos\theta}$, > 0

In our case $p = R_E$ (corresponding to $\theta = \frac{\pi}{2}$). The elevation is given by

$$\tan\alpha = \left(\frac{dr}{r\,d\theta}\right)_{\theta=\pi/2} = \left(\frac{e\,\sin\theta}{(1+e\,\cos\theta)^2}\right)_{\theta=\pi/2} = e$$

and, in turn, the eccentricity follows from (at $\theta = \pi$)

$$R_{GEO} = \frac{R_E}{1-e} \qquad\qquad e = 1 - \frac{R_E}{R_{GEO}}$$

and so $\tan\alpha = 1 - \frac{R_E}{R_{GEO}} = 0.849 \quad ; \quad \alpha = 40.3^{0}$ (3)

The angular momentum (per unit mass) is $h = \sqrt{\mu p} = \sqrt{\mu R_E}$.

Equating this to $R_E\,V_1\,\cos\alpha$,

$$V_1 \cos\alpha = \sqrt{\frac{\mu}{R_E}} \tag{4}$$

(i.e., the horizontal projection of the launch velocity is the local orbital speed, for any apogee radius, R_{GEO} in this case)

Combining (3) and (4), $V_1 = \sqrt{\frac{\mu}{R_E}\left[1+\left(1-\frac{R_E}{R_{GEO}}\right)^2\right]}$ (5)

and this can now be substituted in (2):

$$\Delta V_1 = \sqrt{\frac{\mu}{R_E}\left[1+\left(1-\frac{R_E}{R_{GEO}}\right)^2\right] + v_R^{\,2} - 2v_R\sqrt{\frac{\mu}{R_E}}}$$

$$\boxed{\Delta V_1 = \sqrt{\left(\sqrt{\frac{\mu}{R_E}} - v_R\right)^2 + \frac{\mu}{R_E}\left(1-\frac{R_E}{R_{GEO}}\right)^2}} \tag{6}$$

Upon arrival at I, there will have to be a second burn that will simultaneous accelerate the rocket to $v_{GEO} = \sqrt{\frac{\mu}{R_{GEO}}}$, and rotate the plane to equatorial ($\Delta i = L$).

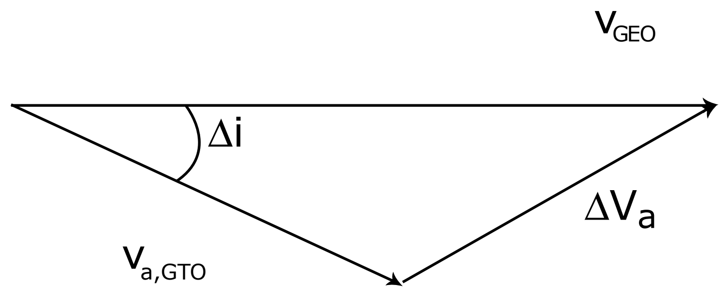


Fig. 3

The apogee velocity is $v_{a,GTO}$, given by

$$R_{GEO}\, v_{a,GTO} = \left(V_1 \cos\alpha\right) R_E = \sqrt{\mu R_E} \qquad (7)$$

and so $\Delta V_a = \sqrt{v_{GEO}^2 + v_{a,GTO}^2 - 2 v_{GEO} v_{a,GTO} \cos \Delta i}$

$$\boxed{\Delta V_a = \sqrt{\frac{\mu}{R_{GEO}}}\sqrt{1 + \frac{R_E}{R_{GEO}} - 2\frac{R_E}{R_{GEO}} \cos L}} \qquad (8)$$

This second burn is probably provided by the spacecraft itself, or else by the launcher's upper stage.

IDEALIZED TWO - BURN GTO INJECTION

One difficulty with the direct injection scheme is the fact that GEO insertion at I must occur on the first pass, because the GTO perigee is actually below the Earth's surface (see Fig. 2). Most operators prefer a temporary parking of the spacecraft in a GTO orbit which has a perigee above the ground, so as to make functional tests and adjustments prior to the final apogee burn (over a period of 2-4 weeks). A modification of the launch sequence to accommodate this is:

(1) Fire Eastwards with α selected for a low apogee ($\sim$ 200 km above ground) at the equatorial crossing.
(2) Fire again at equatorial crossing to raise the apogee to R_{GEO} (no plane change)
(3) At one of the apogee passes, perform the final (circularization + plane change burn).

The formulation is very similar to the previous case.
The elevation α is now given by

$$\tan\alpha = 1 - \frac{R_E}{R_p} \tag{9}$$

(R_p = perigee radius $\simeq R_E + 200\,\text{km}$).

This gives a very shallow trajectory, which is unrealistic; but it is a fair approximation to a real high-elevation launch, followed by a rapid rotation during the rocket firing. For $R_P - R_\epsilon = 200\,\text{km}$, $\alpha = 1.74^0$.

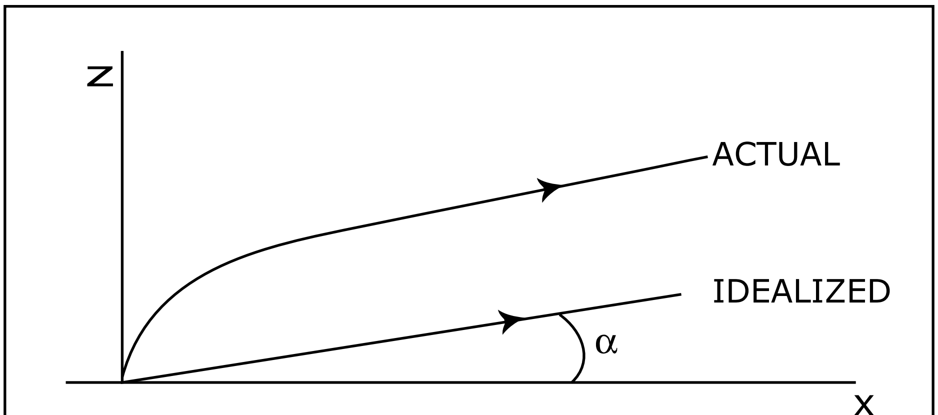


Fig. 4

Eqs. (5) and (6) still hold, with the quality R_{GEO} replaced by R_p, and so

$$\Delta V_1 = \sqrt{\left(\sqrt{\frac{\mu}{R_E}} - v_R\right)^2 + \frac{\mu}{R_E}\left(1 - \frac{R_E}{R_P}\right)^2} \tag{10}$$

which is now smaller, since we are going to a much lower apogee (at r_p).

At this apogee (at the equatorial crossing), we have, as in Eq. (7),

$$v_a = \frac{\sqrt{\mu R_E}}{R_p} \tag{11}$$

and we next need to effect a second rocket firing that will increase velocity to that for the GTO perigee:

$$v_{P_{GTO}} = \sqrt{\frac{\mu}{R_p}\frac{2R_{GEO}}{R_p + R_{GEO}}} \tag{12}$$

No plane change is involved yet, so

$$\Delta V_2 = \sqrt{\frac{\mu}{R_p}}\left[\sqrt{\frac{2R_{GEO}}{R_p + R_{GEO}}} - \sqrt{\frac{R_E}{R_P}}\right] \tag{13}$$

This places the spacecraft on an elliptical GTO orbit, still in the original plane, with apogee at R_{GEO}. The speed at this apogee is:

$$v_{a,GTO} = \sqrt{\frac{\mu}{R_{GEO}}\,\frac{2R_P}{R_P + R_{GEO}}} \tag{14}$$

and so,

$$\Delta V_a = \sqrt{v_{GEO}{}^2 + v_{a,GTO}^2 - 2v_{GEO}\, v_{a,GTO} \cos L}$$

$$\Delta V_a = \sqrt{\frac{\mu}{R_{GEO}} + \frac{\mu}{R_{GEO}}\frac{2R_P}{R_P + R_{GEO}} - 2\frac{\mu}{R_{GEO}}\sqrt{\frac{2R_P}{R_P + R_{GEO}}}\cos L}$$

$$\Delta V_a = \sqrt{\frac{\mu}{R_{GEO}}}\sqrt{1 + \frac{2R_P}{R_P + R_{GEO}} - 2\sqrt{\frac{2R_P}{R_P + R_{GEO}}}\ \cos L} \tag{15}$$

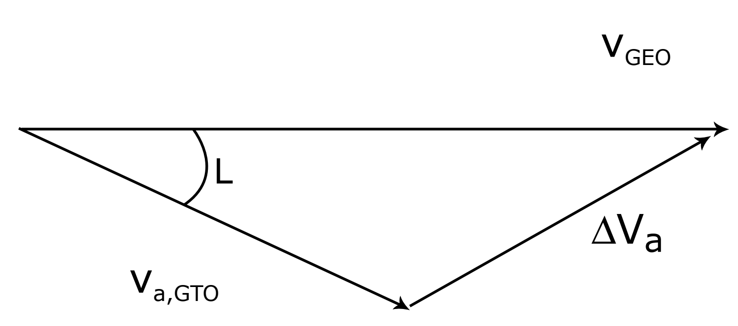


Fig. 5

Some numerical comparisons

We will illustrate these ΔV's by considering launches to GEO from two different locations:

(1) Near the Equator, on at the French kouron complex, and
(2) From mid-latitude, as from Café Canoveral ($L = 28.5^0$).

(1) Equatorial Launch

Option (a): Ground to LEO (300 km), plus LEO-GEO Hohman transfer. No plane changes. Launch to the East.

$$\Delta V = \underbrace{\Delta V_1 + \Delta V_2 - V_R}_{\text{To LEO}, \alpha=0} + \underbrace{\Delta V_3}_{\text{GTO injection}} + \underbrace{\Delta V_4}_{\text{GEO circularization}}$$

ΔV = (8084 – 463) + (10,151 – 7725) + (3071 – 1573)

= 7,621 + 2,426 + 1,498 = 11,545 m/s

Notice this is more than to Escape from mean Earth ($\Delta V \simeq 11,200\,m/s$)
Option (b): Direct injection into GTO from ground

$$\Delta V = \underbrace{\Delta V_1}_{\substack{\alpha=0 \text{ launch to } R=42,200\text{km} \\ (-463\,m/s \text{ for rotation})}} + \underbrace{\Delta V_2}_{\text{GEO circularization}}$$

= (10,420 – 463) + (3071 – 1573) = 9,957 + 1,498 = 11,455 m/s

(2) Launch from L = 28.5º. Launch to East, $\upsilon_R = 407\,m/s$

Option (a): Direct injection to GTO, circularization + plane change at GEO. 2 firings,

$$\Delta V = \underbrace{\Delta V_1}_{\text{Launch with } \alpha=40.3^0} + \underbrace{\Delta V_2}_{\substack{\text{GEO circularization} \\ \text{and plane change}}}$$

= 10,070 + 2,102 = 12,172 m/s

Note the two penalizations for latitude: the elevated launch increased ΔV_1, and the plane change at GEO increases ΔV_2.

Option (b) Direct injection with 3 firings (LEO at 300km)

$$\Delta V = \underbrace{\Delta V_1}_{\text{Launch to a 300 km apogee}} + \underbrace{\Delta V_2}_{\text{Firing to raise apogee to GEO}} + \underbrace{\Delta V_3}_{\substack{\text{Circularization} \\ +\text{Plane change}}}$$

= 7,512 + 2,605 + 1,830 = 11,947 m/s

Is it true that plane change should be all done at end of GTO?

Actually, a small turning combined with initial ΔV_1 (say, from LEO) costs very little ΔV loss, even though V is then large. Try splitting into a Δi_1 and $\Delta i_2 = \Delta i - \Delta i_1$

$$\left.\begin{aligned} \Delta V_1 &= \sqrt{v_{c_1}^2 + v_{p_{GTO}}^2 - 2 v_{c_1} v_{p_{GTO}} \cos \Delta i_1} \\ \Delta V_2 &= \sqrt{v_{c_2}^2 + v_{p_{GTO}}^2 - 2 v_{c_2} v_{a_{GTO}} \cos\left(\Delta i - \Delta i_1\right)} \end{aligned}\right\} \quad \Delta V = \Delta V_1 + \Delta V_2$$

$$\frac{d\Delta V}{d\Delta i_1} = \frac{+\cancel{2} v_{c_1} V_P \sin \Delta i_1}{\cancel{2}\sqrt{v_{c_1}{}^2 + V_P{}^2 - 2 v_{c_1} V_P \cos \Delta i_1}} - \frac{+\cancel{2} v_{c_2} V_a \sin\left(\Delta i - \Delta i_1\right)}{\cancel{2}\sqrt{v_{c_2}{}^2 + V_a{}^2 - 2 v_{c_2} V_a \cos\left(\Delta i - \Delta i_1\right)}} = 0$$

$$v_{c_1} = \sqrt{\frac{\mu}{R_1}}, \qquad v_{c_2} = \sqrt{\frac{\mu}{R_2}}, \qquad v_p = \sqrt{\frac{\mu}{R_1}\frac{2R_2}{R_1+R_2}}, \qquad v_a = \sqrt{\frac{\mu}{R_2}\frac{2R_1}{R_1+R_2}}$$

Call $\rho = \dfrac{R_2}{R_1}$

$$\frac{\sqrt{2\frac{\rho}{1+\rho}}\sin \Delta i_1}{\sqrt{1 + \frac{2\rho}{1+\rho} - 2\sqrt{\frac{2\rho}{1+\rho}}\cos \Delta i_1}} = \frac{\frac{1}{\sqrt{\rho}}\sqrt{\frac{1}{\rho}\frac{2}{1+\rho}}\sin\left(\Delta i - \Delta i_1\right)}{\sqrt{\frac{1}{\rho} + \frac{1}{\rho}\frac{2}{1+\rho} - \frac{2}{\sqrt{\rho}}\sqrt{\frac{1}{\rho}\frac{2}{1+\rho}}\cos\left(\Delta i - \Delta i_1\right)}}$$

$$\frac{\cancel{2}\cancel{\rho}}{\cancel{1+\rho}}\sin^2 \Delta i \ \frac{1}{\cancel{\rho}}\left[1 + \frac{2}{1+\rho} - 2\sqrt{\frac{2}{1+\rho}}\cos\left(\Delta i - \Delta i_1\right)\right] = \frac{1}{\rho^2}\frac{\cancel{2}}{\cancel{1+\rho}}\sin^2\left(\Delta i - \Delta i_1\right)\left[1 + \frac{2\rho}{1+\rho} - 2\sqrt{\frac{2\rho}{1+\rho}}\cos\Delta i\right]$$

$$\rho = \frac{42200}{6370+500} = 6.14265 \qquad\qquad \sqrt{\frac{2\rho}{1+\rho}} = 1.31148$$

$$\frac{1.31148 \,\mathrm{Sin}\, \Delta i_1}{\sqrt{1 + 1.71999 - 2 \times 1.31148 \,\mathrm{Cos}\, \Delta i_1}} = \frac{\frac{0.52916}{6.14265}\mathrm{Sin}\left(28.5 - \Delta i_1\right)}{\frac{1}{\sqrt{6.14265}}\sqrt{1 + 0.28001 - 2 \times 0.52916 \,\mathrm{Cos}\left(28.5 - \Delta i_1\right)}}$$

$$\frac{\text{Sin}\,\Delta i_1}{\sqrt{2.71999 - 2.62296\,\textit{Cos}\,\Delta i_1}} = \frac{0.16280\,\text{Sin}\left(28.5 - \Delta i_1\right)}{\sqrt{1.28001 - 1.05832\,\text{Cos}\left(28.5 - \Delta i_1\right)}}$$

$\boxed{\Delta i_1 = 2.26^0 \text{ optimum}}$ $\Delta i_2 = 26.24^0$

$$\left(\frac{\Delta V}{v_{c_1}}\right)_{op} = \sqrt{1 + \frac{2\rho}{1+\rho} - 2\frac{2\rho}{1+\rho}\cos\Delta i_1} + \sqrt{\frac{1}{\rho} + \frac{1}{\rho}\frac{2}{1+\rho} - \frac{2}{\sqrt{\rho}}\sqrt{\frac{2}{\rho\left(1+\rho\right)}}\cos\Delta i_2}$$

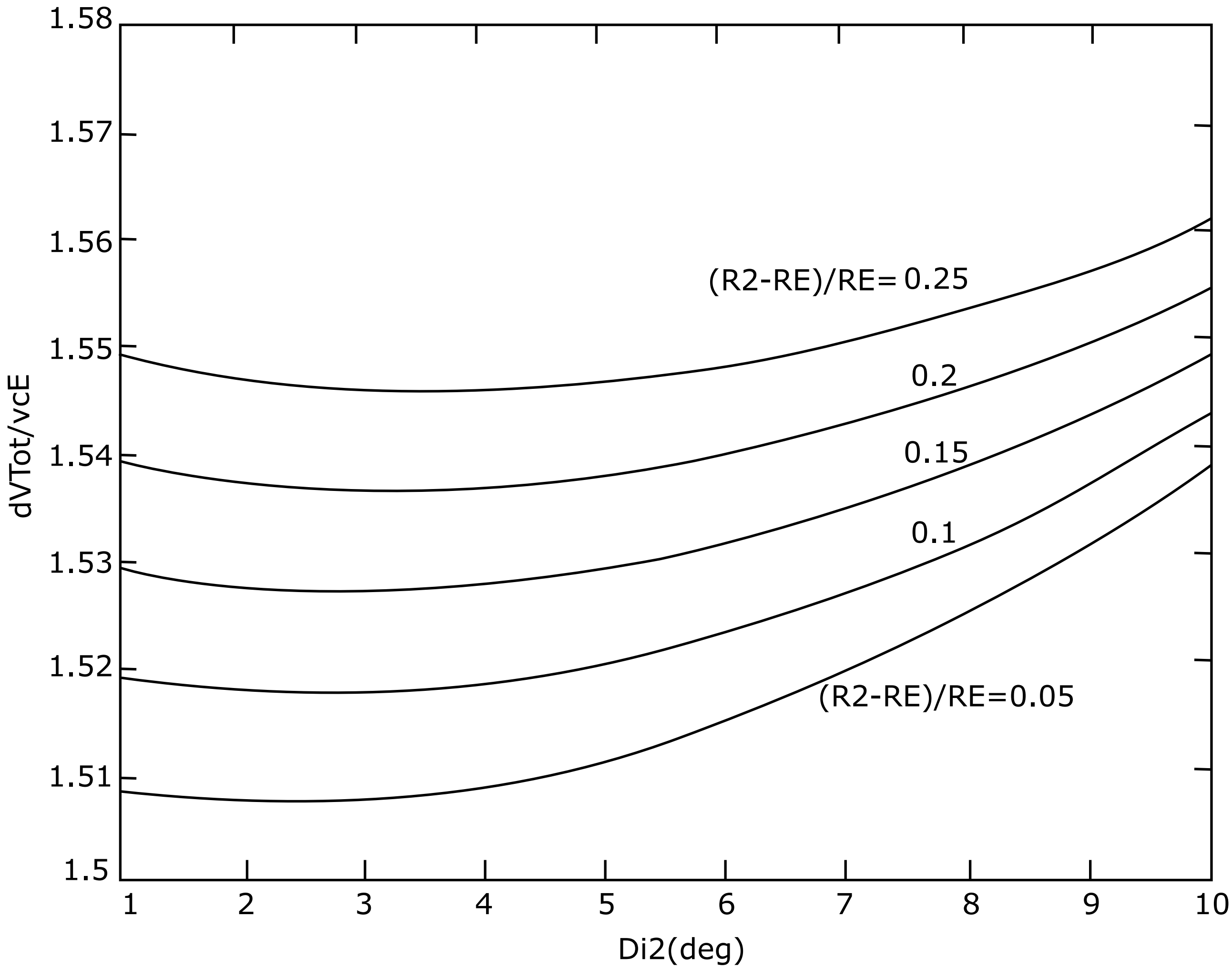
$$\left(\frac{\Delta V}{v_{c_1}}\right)_{op} = \sqrt{2.71199 - 2.62296\cos\Delta i_1} + \frac{1}{\sqrt{6.14265}}\sqrt{1.21001 - 1.05832\cos\Delta i_2}$$

$$= 0.30178 + 0.23227 = \boxed{0.53405}$$ - small improvement

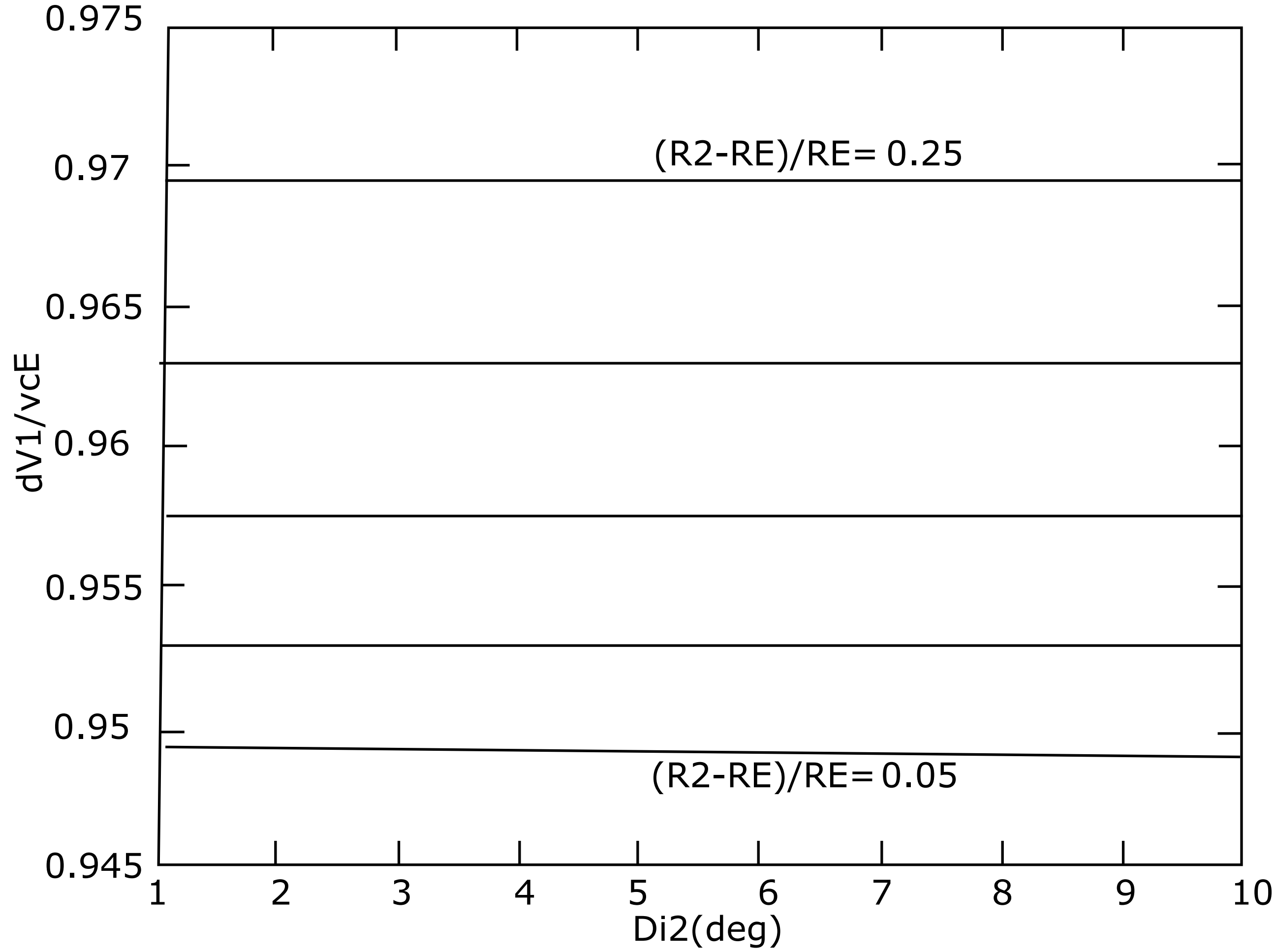
Compare to same with $\Delta i_1 = 0$

$$\left(\frac{\Delta V}{v_{c_1}}\right)_{ref} = 0.29838 + 0.23868 = \boxed{0.53706}$$ - small improvement

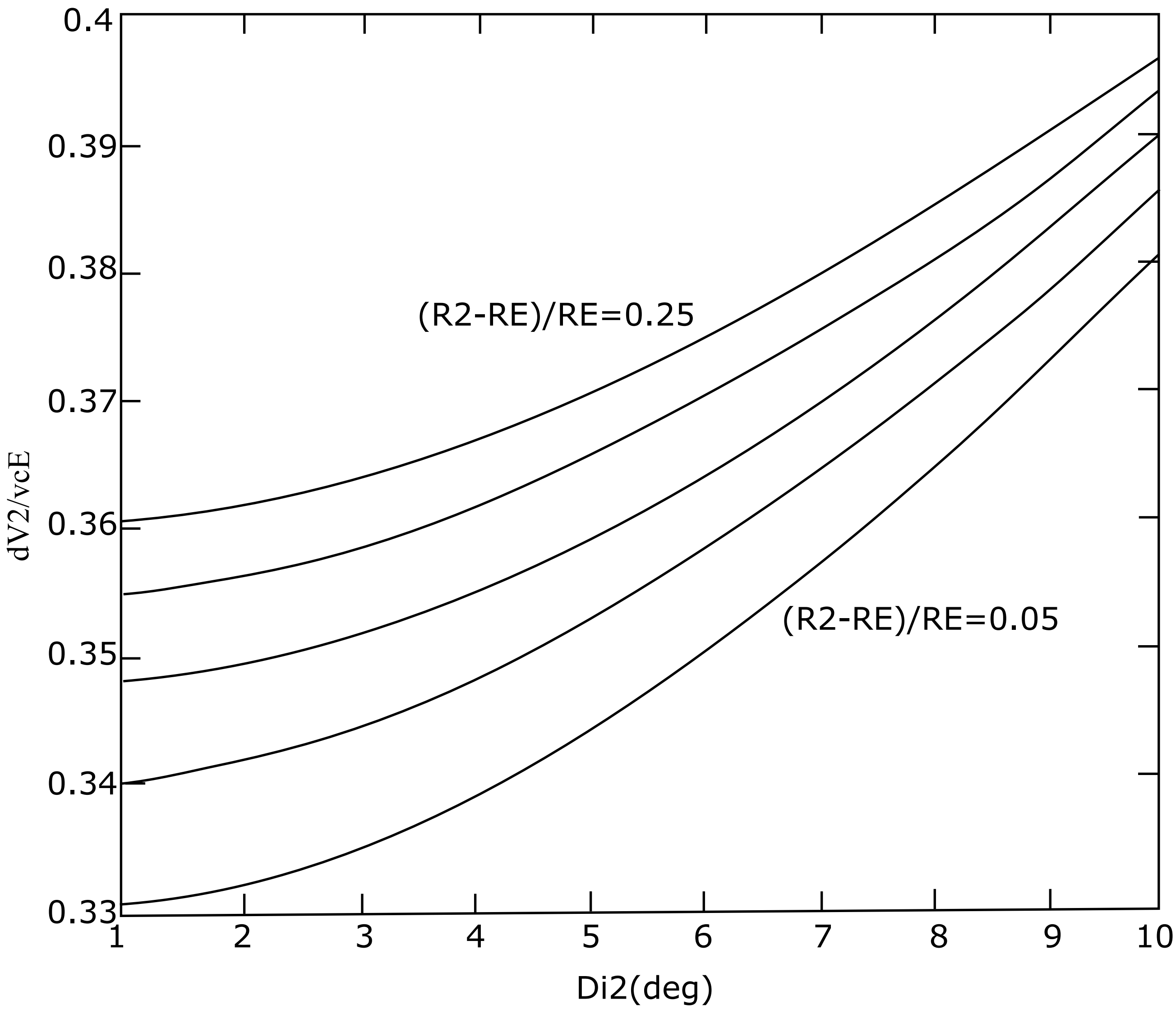
Example: Effects of doing a small plane change Δi_2 simultaneous with the second (apogee-raising) firing in a 3-impulse direct GTO injection.



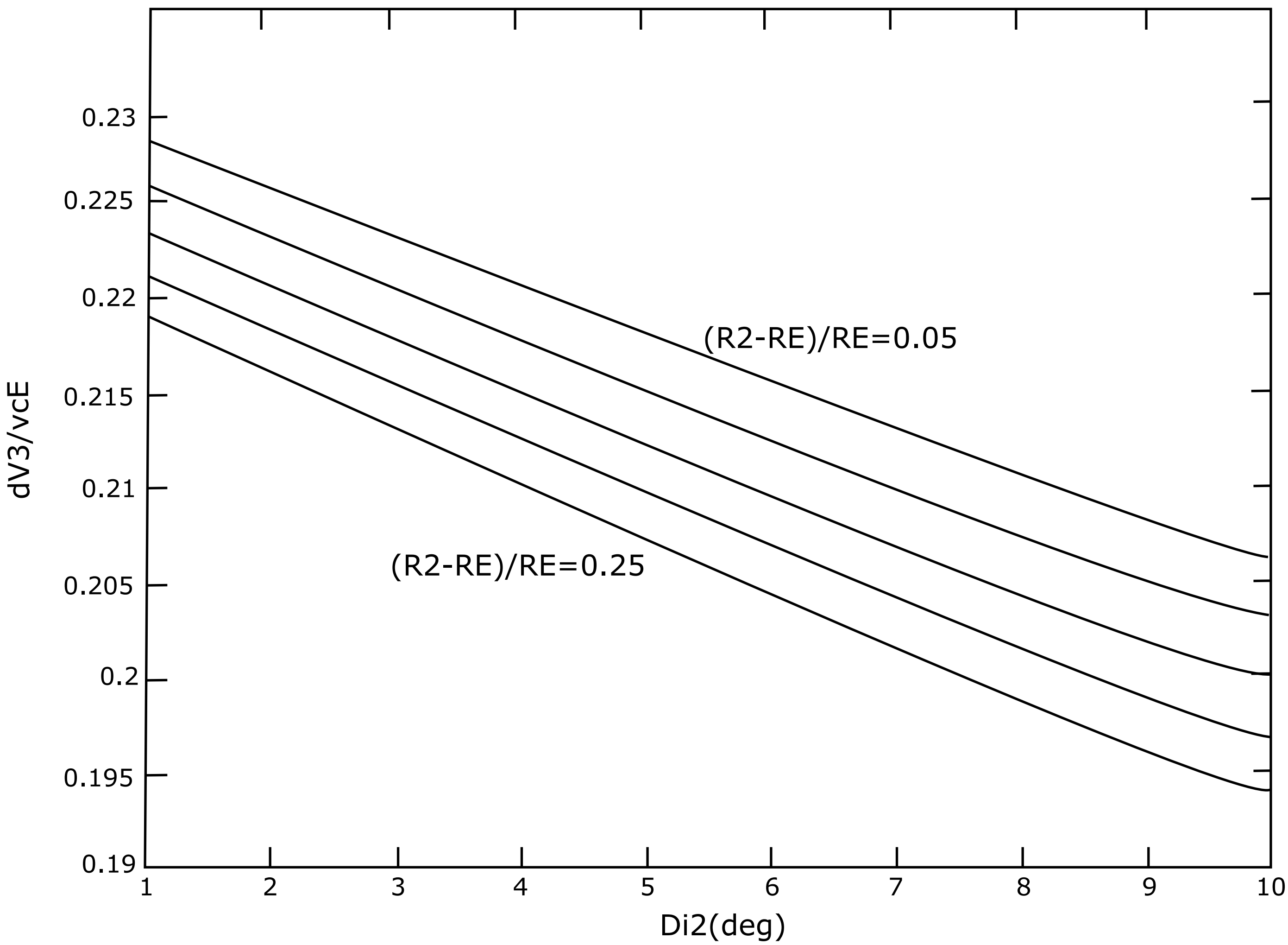
Total dV for three-impulse launch from L=28.5 deg to GEO. Here vcE =sqrt(mu/RE)



dV1 for three-impulse launch from L=28.5 deg to GEO. Here vcE=sqrt(mu/RE)



dV1 for three-impulse launch from L=28.5 deg to GEO. Here vcE=sqrt(mu/RE)



dV3 for three-impulse launch from L=28.5 deg.to GEO. Here, vcE=sqrt(mu/RE)