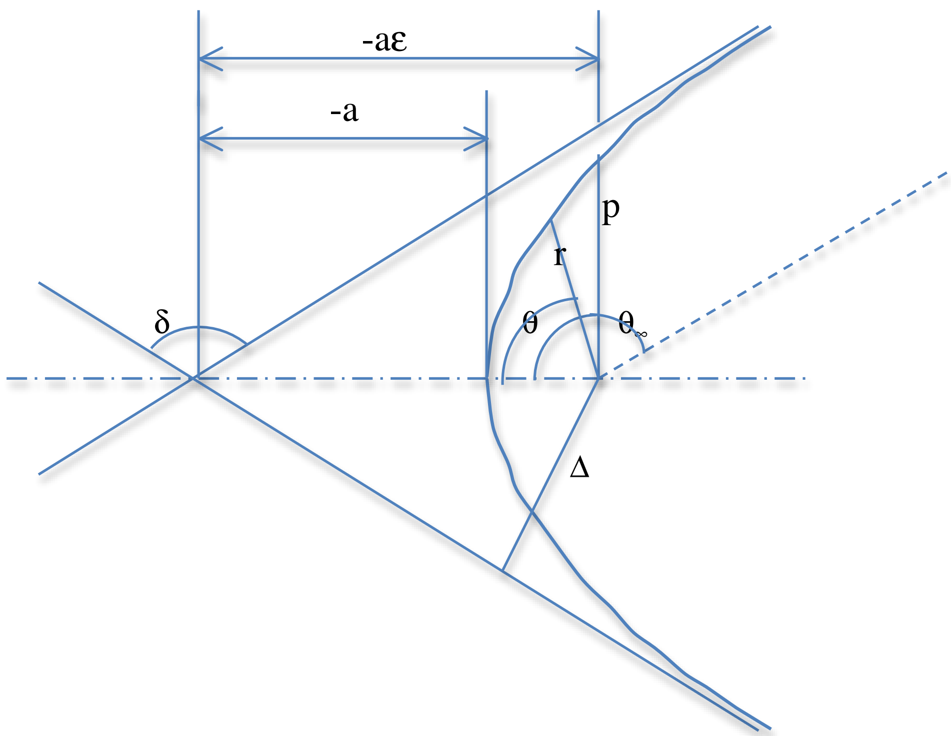
16.50 Lecture 4

Subjects: Hyperbolic orbits. Interplanetary transfer.

(1) Hyperbolic orbits

The trajectory is still described by $r = \dfrac{p}{1+\varepsilon\cos\theta}$, but now we have ε>1, so that the radius tends to infinity at the asymptotic angle $\theta_\infty = \pi - \cos^{-1}(1/\varepsilon)$.



The "parameter" p still has the geometrical significance indicated in the figure, and is therefore a positive number. It is still related to a and ε through $p = a(1-\varepsilon^2)$, but now a is a negative number, so it is (-a) that has a geometrical significance, as indicated in the figure. Note also tat ε is still defined as the ratio of the distance from periapsis to center to the distance from focus to center.

The energy is still given by $E = \dfrac{1}{2}v^2 - \dfrac{\mu}{r} = -\dfrac{\mu}{2a}$, and is now positive. The angular momentum is still given by $h = r^2\dot{\theta} = \sqrt{\mu p} = \sqrt{a(1-\varepsilon^2)}$.

There are a few new parameters of interest in this case:

The trajectory deflection, $$\delta = \pi - 2(\pi - \theta_\infty) = \pi - 2\cos^{-1}\frac{1}{\varepsilon} = 2\sin^{-1}\frac{1}{\varepsilon}$$

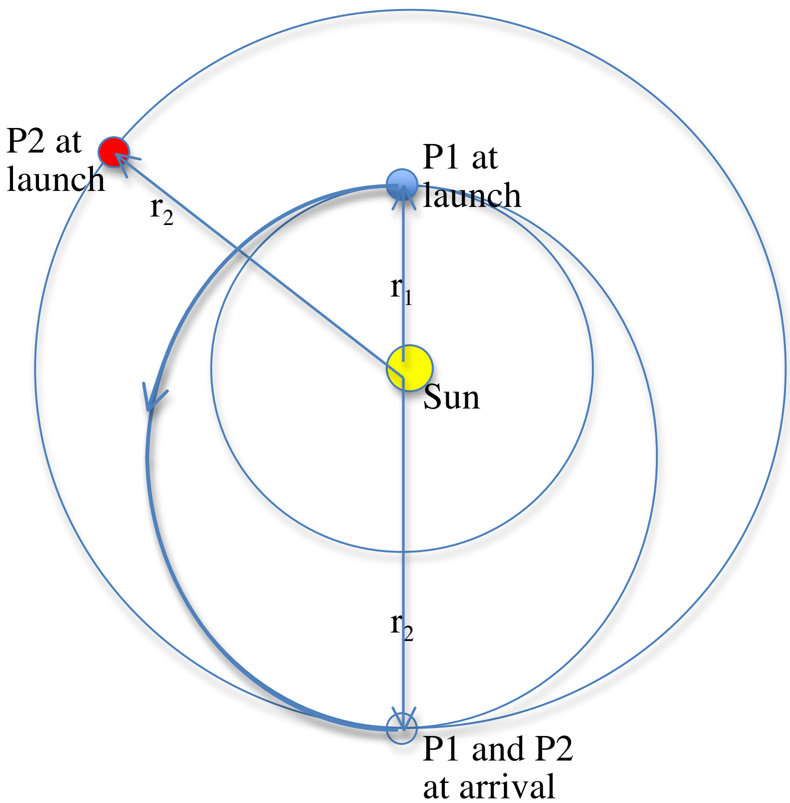
The miss distance
$$\Delta = -a\varepsilon \sin(\pi - \theta_\infty) = -a\varepsilon \sin\theta_\infty = \frac{p}{\sqrt{\varepsilon^2 - 1}}$$

The excess hyperbolic velocity,
$$v_\infty = \sqrt{2E} = \sqrt{\frac{\mu}{(-a)}}$$

In the specialized technical literature, the term "c_3" is often used, meaning simply v_∞^2.

2) Interplanetary transfer

We assume for the moment that our craft has "escaped the field of planet 1" (meaning it is outside its sphere of influence), and so may be considered to be in orbit about the Sun. In order for it to reach planet 2, its orbit about the Sun must intersect that of planet 2.



Assume that the planetary orbits are circular. Then it is clear that the trajectory of least energy which will allow the transfer is that which is just tangent to the orbits of the home and target planets; this is called the Hohmann transfer orbit, which is the half-ellipse that is sketched in the figure. The Heliocentric velocity at the start of this Hohman arc is the periapsis (perihelion in this case) velocity, as described at the end of the last lecture:

$$v_{p1} = \sqrt{\frac{\mu_S}{r_1} \frac{2r_2}{r_1 + r_2}}$$

so that if we launch the ship in the direction of motion of the planet, it must have a relative velocity

$$v_{rel,1} = \sqrt{\frac{\mu_S}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)$$

with respect to planet 1 <u>after escape</u> from the planet. By definition, this is the "excess hyperbolic velocity" relative to the planet, $v_{\infty 1}$, and the total energy relative to planet 1 at he edge of the sphere of influence is simply $\frac{1}{2}(v_{\infty 1})^2$.

Suppose the launch was for the surface of planet 1 (radius R_1), and ignore its rotation. Just after launch, during which the rocket has imparted an instantaneous velocity increment ΔV_1, the energy per unit mass (relative to the planet) is $\frac{1}{2}(\Delta V_1)^2 - \frac{\mu_1}{R_1}$, and this must be the same as $\frac{1}{2}(v_{\infty 1})^2$, by energy conservation with respect to planet 1 <u>inside the sphere of influence</u>. We then have

$$\frac{1}{2}(\Delta V_1)^2 - \frac{\mu_1}{R_1} = \frac{1}{2}v_{\infty 1}^2 = \frac{1}{2}\frac{\mu_S}{r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)^2$$

from which the first delta-V delivered by the rockets must be

$$\Delta V_1 = \sqrt{\frac{2\mu_1}{R_1} + \frac{\mu_S}{r_1}\left(\sqrt{\frac{2r_2}{r_1+r_2}}-1\right)^2}$$

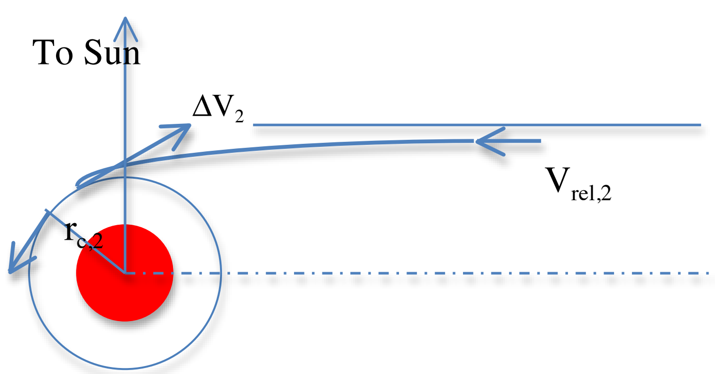
The procedure is similar when considering the approach to planet 2. The spacecraft will have then a heliocentric velocity equal to the apoaxis (apohelion) velocity

$$v_{a2} = \sqrt{\frac{\mu_S}{r_2}\frac{2r_1}{r_1+r_2}}$$

and a relative velocity with respect to the planet

$$v_{rel,2} = \sqrt{\frac{\mu_S}{r_2}}\left(1-\sqrt{\frac{2r_1}{r_1+r_2}}\right)$$

which is also the excess hyperbolic velocity with respect to planet 2. It is worth noting a this point that the spacecraft heliocentric velocity is less than that of the planet itself, so that, as seen from the planet, the spacecraft will be approaching from its advancing side. For capture into a circular orbit of radius R_{c2}, the geometry is shown below:



Just before the insertion rocket firing, the energy per unit mass relative to planet 2 is equal to ½ $(v_{rel,2})^2$, and it is also equal to the sum of the kinetic energy at that point of closest approach, plus the potential energy: $\frac{1}{2}(v_{closest\ app.})^2 - \frac{\mu_2}{r_{c2}}$. Thus we must have

$$v_{closest\ app.} = \sqrt{2\frac{\mu_2}{r_{c2}} + \frac{\mu_S}{r_2}(1-\sqrt{\frac{2r_1}{r_1+r_2}})^2}$$

and the insertion velocity increment must be this, minus the orbital velocity around planet 2:

$$\Delta V_2 = \sqrt{2\frac{\mu_2}{r_{c2}} + \frac{\mu_S}{r_2}(1-\sqrt{\frac{2r_1}{r_1+r_2}})^2} - \sqrt{\frac{\mu_2}{r_{c2}}}$$

Comparison to simple Escape+Transfer+Capture.

A simple-minded approach to the same mission would be to first apply an impulse at the surface of planet 1 to achieve escape ($\Delta V_{esc,1} = \sqrt{2\mu_1 / R_1}$), then, after slowing down to zero velocity with respect to planet 1, apply a second impulse to enter the elliptic transfer orbit towards planet 2 (this would be our $v_{rel,1}$), then, in the vicinity (but still outside the SOI of) planet 2, apply a third impulse to match the heliocentric velocity of planet 2 (this would be our $v_{rel,2}$), and finally, starting from zero relative velocity "far" from planet 2, apply a fourth impulse to capture the craft into orbit about planet 2 (this is equal to the escape velocity from a distance R_2 to the planet, $\Delta V_{esc,2} = \sqrt{2\mu_2 / R_2}$). You can easily check that the two impulses we derived before are, respectively,

$$\Delta V_1 = \sqrt{(\Delta V_{esc,1})^2 + (v_{rel,1})^2}$$
$$\Delta V_2 = \sqrt{(\Delta V_{esc,2})^2 + (v_{rel,2})^2}$$

and so our previous scheme is definitely more effective. These two strategies are called sometimes the Hohmann (simple, four impulses) and the Oberth (combined, two impulses) maneuvers.