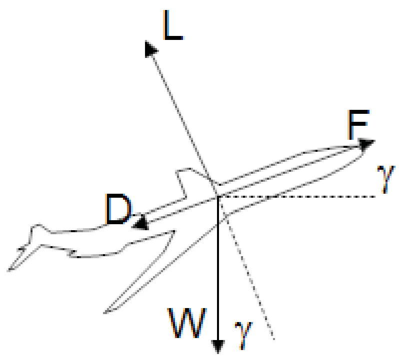
Homework 6: Off-Design Performance of Small Turboprop Engine

a) At the end of climb, $z = 6000\ m$, we have $T_0 = 261K\ (a_0 = 323.8\ m/s)$, and $P_0 = 4.86 * 10^4\ Pa$. Also, $M_0 = 0.6\ (\theta_0 = 1 + 0.2 * 0.6^2 = 1.072, T_{t0} = \theta_0 T_0 = 279.8K)$. Finally,

$P_{t0} = P_0 \theta_0^{\frac{\gamma}{\gamma-1}} = 6.199 * 10^4\ Pa.$



Force balance along and transverse to the trajectory:

$L = W \cos\gamma$ (1)

$F = D + W \sin\gamma$ (2)

From equation (2), $F = \frac{L}{L/D} + W \sin\gamma = W\left(\frac{\cos\gamma}{L/D} + \sin\gamma\right)$

We are given $W = 2000 * 9.8 = 19{,}600\ N, \gamma = 20°, L/D = 15$, so

$$F = 19.600\left(\frac{\cos 20°}{15} + \sin 20°\right) = 7{,}931\ N$$

b) Since $T_{t4} = 1200\ K$, we have $\theta_t = \frac{1200}{261} = 4.598.$

The compressor ratio is chosen for maximum thrust, so

$\tau_c = \frac{\sqrt{\theta_t}}{\theta_0} = \frac{\sqrt{9.598}}{1.072} = 2.0003$ (3)

$T_{t3} = T_{t0}\tau_c = 559.7\ K$ (4)

$\pi_c = \tau_c^{\frac{\gamma}{\gamma-1}} = (2.0003)^{3.5} = 11.32$ (5)

$P_{t3} = P_{t0}\pi_c = 7.017 * 10^5\ Pa$ (6)

From the shaft power balance:

$\tau_t = 1 - \frac{\tau_c - 1}{\theta_t}\theta_0 = 1 - \frac{1.0003}{4.598}(1.072) = 0.7668$

$\pi_t = \tau_t^{\frac{\gamma}{\gamma-1}} = 0.3948$

From these results:

$T_{t5} = \tau_t T_{t4} = 0.7668 * 1200 = 920.2\ K$
$P_{t5} = \pi_t P_{t4} = \pi_t P_{t3} = 0.3948 * 7.017 * 10^5 = 2.770 * 10^5\ Pa$
$T_{t7} = T_{t5}$
$P_{t7} = P_{t5}$

c) To calculate the air flow rate $\dot{m}$ we need the value of $\frac{F}{\dot{m}a_0}$. We assume here *<u>matched</u>* *<u>exhaust</u>* conditions and use:

$$\frac{F}{\dot{m}a_0} = \sqrt{\frac{2}{\gamma-1}(\theta_0\tau_c\tau_t - 1)\frac{\theta_t}{\theta_0\tau_c}} - M_0 \qquad (7)$$

$$\frac{F}{\dot{m}a_0} = \sqrt{5(1.072 * 2000 * 0.7668 - 1)\frac{4.598}{1.072*2000}} - 0.6 = 2.0278$$

$$\dot{m} = \frac{F}{2.0278a_0} = \frac{7931}{2.0278*323.8} = 12.078\ kg/s$$

d) The general flow rate expression is:

$$\dot{m} = \overline{m}\Gamma\frac{P_tA}{\sqrt{RT_t}} \qquad (8)$$

We have:

$$\Gamma = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.6339 \qquad (9)$$

$$R = \frac{8.314}{0.0289} = 287\ {}^{J}/_{kg * K} \qquad (10)$$

Applying this at the *<u>choked</u>* stations 4 and 7:

$$A_4 = \dot{m}\frac{\sqrt{RT_{t4}}}{\Gamma P_{t4}} \qquad (11)$$

$$A_4 = 12.08 * \frac{\sqrt{287*1200}}{0.6339*7.017*10^5} = 0.01594\ m^2$$

$$A_7 = \dot{m}\frac{\sqrt{RT_{t7}}}{\Gamma P_{t7}} \qquad (12)$$

$$A_7 = 12.08 * \frac{\sqrt{287*920.2}}{0.6339*2.77*10^5} = 0.03535\ m^2$$

$$D_7 = \sqrt{\frac{4}{\pi}A_7} = 0.2122\ m^2 \qquad (13)$$

For station 2:

$$\overline{m}_2 = M_2\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M_2^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.4\left(\frac{1.2}{1+0.2*0.4^2}\right)^3 = 0.6289 \qquad (14)$$

$$A_2 = \frac{\dot{m}}{\overline{m}_2}\frac{\sqrt{RT_{t0}}}{\Gamma P_{t0}} = \frac{12.08}{0.6289}\frac{\sqrt{287*279.8}}{0.6339*6.166*10^4} = 0.1385\ m^2 \qquad (15)$$

$$D_2 = \sqrt{\frac{4}{\pi}A_2} = 0.420\ m \qquad (16)$$

e) The combustion energy balance gives:

$$f = \frac{c_p(T_{t4} - T_{t3})}{h} \qquad (17)$$

$$f = \frac{1005(1200 - 559.7)}{43*10^6} = 0.01497$$

Therefore:

$$\dot{m}_f = f\dot{m} = 0.01497 * 12.08 = 0.1808\ kg/s \qquad (18)$$

The specific impulse is:

$$I = \frac{F}{\dot{m}_f g} = \frac{7.931}{0.1808*9.8} = 4.477\ s \qquad (19)$$

f) The flight speed is $u_0 = M_0 a_0 = 0.6 * 323.3 = 194.3\ m/s$. **The exhaust velocity is then:**

$$u_e = u_0 + \frac{F}{\dot{m}} = 194.3 + \frac{7931}{12.08} = 850.9\ m/s \qquad (20)$$

The propulsive efficiency is then:

$$\eta_p = \frac{2u_0}{u_0 + u_e} = \frac{2*194.3}{194.3*850.9} = 0.3718 \qquad (21)$$

This result is not a very high propulsive efficiency.

The overall efficiency follows from the specific impulse:

$$\eta_{ov} = \frac{gu_0}{h} I = \frac{9.8*194.3}{43*10^6} 4477 = 0.1983 \qquad (22)$$

The thermodynamic efficiency is then:

$$\eta_{th} = \frac{\eta_{ov}}{\eta_p} = 0.5331 \qquad (23)$$

Note: The thermodynamic efficiency can also be calculated directly using equation (24):

$$\eta_{th} = \frac{\frac{1}{2}\dot{m}(u_e^2 - u_0^2)}{\dot{m}_f h} \qquad (24)$$

Compressor Working Line

When conditions change ,the nondimensional flow $\bar{m}_2$ and the compressor ratios τ_c, π_c both change, but they do so in a coordinated way. As explained in lecture 19, we must have:

$$\bar{m}_2(M_2) = \frac{A_4}{A_2}\pi_c \sqrt{\frac{1-\tau_t}{\pi_c^{\frac{\gamma-1}{\gamma}} - 1}} \qquad (25)$$

Where τ_t remains constant. Often $\bar{m}_2$ is reported as the *relative* flow, **normalized by its design value:**

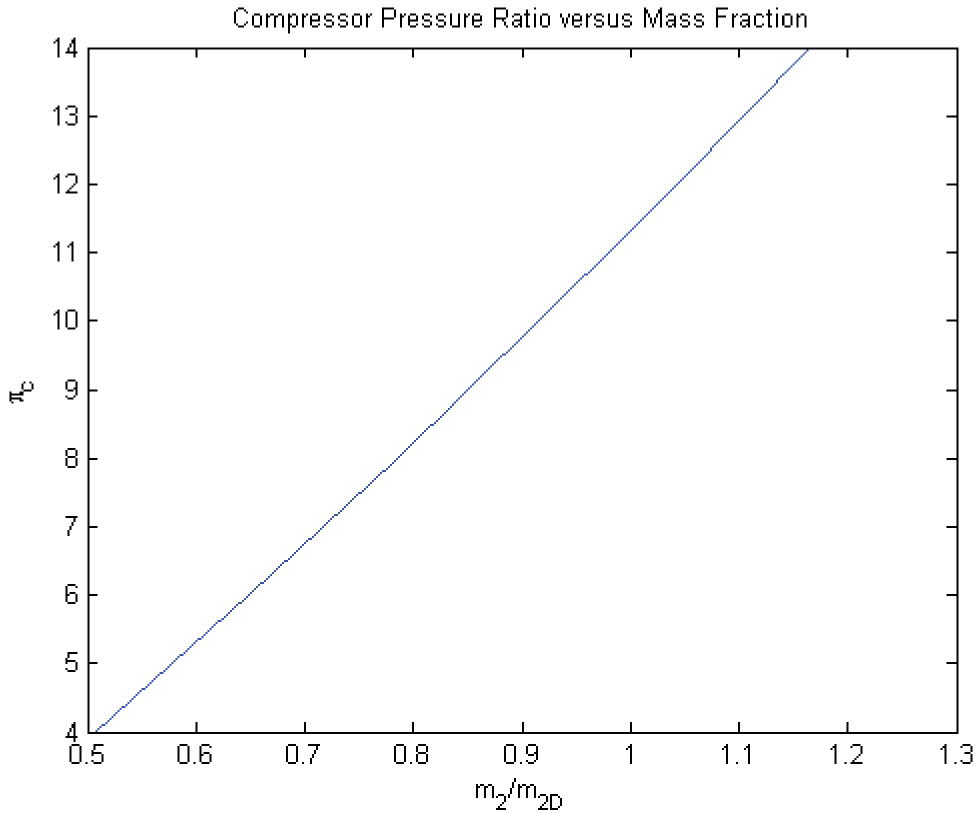
$$\bar{m}_{2D} = \frac{A_4}{A_2}(\pi_c)_D \sqrt{\frac{1-\tau_t}{\pi_{CD}^{\frac{\gamma-1}{\gamma}} - 1}} \qquad (26)$$

Dividing equations (25)/(26):

$$\frac{\bar{m}_2}{\bar{m}_{2D}} = \frac{\pi_c}{\pi_{cD}}\sqrt{\frac{\pi_{cD}^{\frac{\gamma-1}{\gamma}}-1}{\pi_c^{\frac{\gamma-1}{\gamma}}-1}} \qquad (27)$$

Some values are tabulated below, using our result $\pi_{cD} = 11.32$:

π_c	4	6	8	10	11.32	12	14
$\bar{m}_2/\bar{m}_{2D}$	0.5070	0.6484	0.7847	0.9159	1	1.0426	1.1659



Concept Questions

1) If F is doubled, at the same M_0, T_0, M_2, T_{t4}, we will still have *the same* $\tau_c, \tau_t, \pi_c, \pi_t, \bar{m}_2$ and *the same* $T_{t3}, T_{t4}, T_{t5}, P_{t3}, P_{t4}, P_{t5}, I, \eta_p, \eta_{th}, \eta_{ov}$. But the *mass flow rate would be doubled,* as would the flow areas A_2, A_4, A_7.

2) If T_{t4} is raised to 1500 K, just about everything changes. We would get more thrust per unit flow $\frac{F}{\dot{m}a_0}$, and hence *less flow* $\dot{m}$ and smaller cross-sections. The changes in I and in the efficiencies are less clear. The thermodynamic efficiency is that of a Brayton cycle with a pressure ratio $(\theta_0\tau_c)^{\frac{\gamma}{\gamma-1}}$, and so $\eta_{th} = 1 - \frac{1}{\theta_0\tau_c} = 1 - \frac{1}{\sqrt{\theta_t}}$. This means a higher η_{th} when T_{t4} increases. But, since a higher θ_t

implies a higher $\frac{F}{\dot{m}a_0}$, the *propulsive efficiency* η_p will be *less*. $\eta_p = \frac{2}{2+\left(\frac{F/\dot{m}a_0}{M_0}\right)}$. The product of the two η_{ov} also turns out to be *less* at higher M_0, although this is more difficult to see.

So increasing T_{t4} gives more *thrust,* hence a smaller engine, but at the cost of higher fuel consumption.