16.346 Astrodynamics
Fall 2008

Terminology

Disturbing acceleration $\mathbf{a}_d = -c\rho v^2\,\mathbf{i}_t$

Ballistic coefficient c

Atmospheric density

$$\boxed{\rho(r) = \rho_0 \exp\left(-\frac{r-q}{H}\right)} = \rho_0 \exp[-\nu(1-\cos E)] \qquad \text{where} \qquad \boxed{\nu = \frac{ae}{H}}$$

Radius of orbit $r = a(1 - e\cos E)$

Pericenter radius of orbit $q = a(1-e)$

Density at pericenter radius ρ_0

Scale height of atmosphere H

The Variational Equation

$$\frac{de}{dt} = \frac{1}{v}\left[2(e + \cos f)a_{dt} - \frac{r}{a}\sin f\,a_{dn}\right] \implies \boxed{\frac{de}{dt} = \frac{2}{v}(e + \cos f)a_{dt}}$$

$$r = a(1 - e\cos E) = \frac{a(1 - e^2)}{1 + e\cos f}$$

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad \cos E = \frac{e + \cos f}{1 + e\cos f}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) = n^2 a^2 \frac{1 + e\cos E}{1 - e\cos E}$$

Hence

$$\frac{de}{dt} = -2c\rho v(e + \cos f) = -2c \times \rho_0 \exp[-\nu(1-\cos E)] \times na\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \times \frac{p}{r}\cos E$$

so that

$$\boxed{\frac{de}{dt} = -2c\rho_0\frac{pna}{r}e^{-\nu}e^{\nu\cos E}\cos E\sqrt{\frac{1 + e\cos E}{1 - e\cos E}}}$$

Series Representation

Expand in a power series   **See Appendix C**

$$\frac{1 + e\cos E}{1 - e\cos E} = 1 + 2e\cos E + 2e^2\cos^2 E + 2e^3\cos^3 E + \cdots$$

$$\cos E\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} = \cos E(1 + e\cos E + \tfrac{1}{2}e^2\cos^2 E + \tfrac{1}{2}e^3\cos^3 E + \cdots)$$

This can be converted to a Fourier Cosine Series

$$A_0 + A_1\cos E + A_2\cos 2E + A_3\cos 3E + \cdots$$

using Euler's pattern

16.346 Astrodynamics   *Lecture 28*

Prob. 5–11

$$\cos^2 E = \tfrac{1}{2}(\cos 2E + 1)$$
$$\cos^3 E = \tfrac{1}{4}(\cos 3E + 3\cos E)$$
$$\cos^4 E = \tfrac{1}{8}(\cos 4E + 4\cos 2E + 3)$$
$$\cos^5 E = \tfrac{1}{16}(\cos 5E + 5\cos 3E + 10\cos E)$$

$$A_0 = \tfrac{1}{2}e(1 + \tfrac{3}{8}e^2)$$
$$A_1 = 1 + \tfrac{3}{8}e^2 + \tfrac{15}{64}e^4$$
$$A_2 = \tfrac{1}{2}e(1 + \tfrac{1}{2}e^2)$$
$$A_3 = \tfrac{1}{8}e^2(1 + \tfrac{15}{16}e^2)$$

A more meaningfull result is obtained by averaging over a complete orbit

$$\overline{\frac{de}{dt}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{de}{dt}\, dM = \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{a}\frac{de}{dt}\, dE$$

to obtain

$$\boxed{\overline{\frac{de}{dt}} = -2c\rho_0 pne^{-\nu} \sum_{k=0}^{\infty} A_k I_k(\nu)}$$

where

$$\boxed{I_k(\nu) = \frac{1}{\pi}\int_0^{\pi} e^{\nu\cos E}\cos kE\, dE}$$

is the modified Bessel function of the first kind of order k.

Because of the relation to Bessel functions through the identity

$$I_k(\nu) = i^{-k} J_k(i\nu)$$

it is sometimes referred to as a Bessel Function with Imaginary Argument. It can also be expressed as the series expansion

$$I_k(\nu) = \sum_{j=0}^{\infty} \frac{(\tfrac{1}{2}\nu)^{k+2j}}{j!\,(k+j)!}$$

Calculating Modified Bessel Functions

Since ν is generally large, we can use the *asymptotic expansion* to calculate $I_k(\nu)$:

$$e^{-\nu}\sqrt{2\pi\nu}\, I_k(\nu) \sim -\frac{4k^2 - 1^2}{1!\,8\nu} + \frac{(4k^2 - 1^2)(4k^2 - 3^2)}{2!\,(8\nu)^2}$$
$$-\frac{(4k^2 - 1^2)(4k^2 - 3^2)(4k^2 - 5^2)}{3!\,(8\nu)^3} + \cdots$$

as obtained by Carl Gustav Jacob Jacobi in 1849. Although the series will eventually diverge as the number of terms increases, it can be used for numerical computation by employing only those terms whose magnitude decreases as we take more and more terms. The order of magnitude of the error at any stage is equal to the magnitude of the first term omitted.

Note: Leonard Euler was using divergent series for computation in the middle 1770's but the formal theory of asymptotic series was developed much later by Henri Poincaré in 1886.

A series of the form

$$a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots$$

where the a_i's are independent of x, is said to represent the function $f(x)$ asymptotically for large x whenever

$$\lim_{x \to \infty} x^n \left[f(x) - \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} \right) \right] = 0$$

for $n = 0, 1, 2, 3, \ldots$. The series will usually diverge.

We can use the recursion formula

$$e^{-\nu} I_{k+1}(\nu) = e^{-\nu} I_{k-1}(\nu) - \frac{2k}{\nu} e^{-\nu} I_k(\nu)$$

to calculate higher-order values of $e^{-\nu} I_k(\nu)$ starting with $e^{-\nu} I_0(\nu)$ and $e^{-\nu} I_1(\nu)$.

Also we can use the continued fraction

$$\frac{e^{-\nu} I_1(\nu)}{e^{-\nu} I_0(\nu)} = \cfrac{\frac{1}{2}\nu}{1 + \cfrac{(\frac{1}{2}\nu)^2}{2 + \cfrac{(\frac{1}{2}\nu)^2}{3 + \cfrac{(\frac{1}{2}\nu)^2}{4 + \cfrac{(\frac{1}{2}\nu)^2}{5 + \ddots}}}}}$$

to calculate $e^{-\nu} I_1(\nu)$ from $e^{-\nu} I_0(\nu)$.

A sensible receipt would be First: Calculate $e^{-\nu} I_0(\nu)$ from the asymptotic series

$$e^{-\nu} \sqrt{2\pi\nu}\, I_0(\nu) \sim 1 + \frac{1^2}{1!\, 8\nu} + \frac{1^2 \cdot 3^2}{2!\, (8\nu)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{3!\, (8\nu)^3} + \cdots$$

Next: Calculate $e^{-\nu} I_1(\nu)$ using the continued fraction. Higher order functions can then be obtained using the recursion formula.