16.346 Astrodynamics
Fall 2008

The Rotation Matrix # 2.1

Let $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ and $\mathbf{i}_\xi$, $\mathbf{i}_\eta$, $\mathbf{i}_\zeta$ be two sets of orthogonal unit vectors.

$$\begin{aligned}\mathbf{i}_\xi &= l_1\,\mathbf{i}_x + m_1\,\mathbf{i}_y + n_1\,\mathbf{i}_z\\ \mathbf{i}_\eta &= l_2\,\mathbf{i}_x + m_2\,\mathbf{i}_y + n_2\,\mathbf{i}_z\\ \mathbf{i}_\zeta &= l_3\,\mathbf{i}_x + m_3\,\mathbf{i}_y + n_3\,\mathbf{i}_z\end{aligned}$$

where l_1, m_1, $\ldots$, n_3 are called direction cosines.

Any vector $\mathbf{r}$ can be expressed as $\mathbf{r} = x\,\mathbf{i}_x + y\,\mathbf{i}_y + z\,\mathbf{i}_z = \xi\,\mathbf{i}_\xi + \eta\,\mathbf{i}_\eta + \zeta\,\mathbf{i}_\zeta$. Then

$$\boxed{\mathbf{R} = \begin{bmatrix}\mathbf{i}_x\cdot\mathbf{i}_\xi & \mathbf{i}_x\cdot\mathbf{i}_\eta & \mathbf{i}_x\cdot\mathbf{i}_\zeta\\ \mathbf{i}_y\cdot\mathbf{i}_\xi & \mathbf{i}_y\cdot\mathbf{i}_\eta & \mathbf{i}_y\cdot\mathbf{i}_\zeta\\ \mathbf{i}_z\cdot\mathbf{i}_\xi & \mathbf{i}_z\cdot\mathbf{i}_\eta & \mathbf{i}_z\cdot\mathbf{i}_\zeta\end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{bmatrix}}$$

is the rotation matrix and

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \mathbf{R}\begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix}\xi\\ \eta\\ \zeta\end{bmatrix} = \mathbf{R}^{\mathrm{T}}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

Since

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \quad \text{so that} \quad \boxed{\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}}$$

then $\mathbf{R}$ is an orthogonal matrix.

Kinematics in Rotating Coordinates #2.5

Use an asterisk to distinguish a vector resolved along fixed axes from the same vector resolved along the rotating axes:

$$\mathbf{r}^* = \mathbf{R}\mathbf{r} \qquad \mathbf{v}^* = \mathbf{R}\mathbf{v} \qquad \mathbf{a}^* = \mathbf{R}\mathbf{a}$$

where $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ are the position, velocity, and acceleration vectors whose components are understood to be projections along the moving axes.

$$\mathbf{v}^* = \frac{d\mathbf{r}^*}{dt} = \mathbf{R}\left[\frac{d\mathbf{r}}{dt} + \boldsymbol{\Omega}\mathbf{r}\right] = \mathbf{R}\left[\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r}\right] = \mathbf{R}\mathbf{v}$$

where $\boxed{\boldsymbol{\Omega} = \mathbf{R}^{\mathrm{T}}\dfrac{d\mathbf{R}}{dt}} = \begin{bmatrix}0 & -\omega_\zeta & \omega_\eta\\ \omega_\zeta & 0 & -\omega_\xi\\ -\omega_\eta & \omega_\xi & 0\end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix}\omega_\xi\\ \omega_\eta\\ \omega_\zeta\end{bmatrix} \qquad \boxed{\boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}}$

The **angular velocity vector** $\boldsymbol{\omega}$ is identified as the angular velocity of the moving coordinate system with respect to the fixed system.

For the acceleration vector

$$\mathbf{a}^* = \frac{d^2\mathbf{r}^*}{dt^2} = \mathbf{R}\left[\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\Omega}\frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\Omega}}{dt}\mathbf{r} + \boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{r}\right]$$

$$= \mathbf{R}\left[\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\right] = \mathbf{R}\mathbf{a}$$

The four terms which comprise the acceleration referred to rotating axes are called the **Observed**, the **Coriolis**, the **Euler**, and the **Centripetal** accelerations, respectively. (The observed velocity and acceleration vectors $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ will sometimes be denoted by $\mathbf{v}_{rel}$ and $\mathbf{a}_{rel}$ since they are quantities measured *relative* to the rotating axes. The symbols $\mathbf{v}$ and $\mathbf{a}$ will be reserved for the total velocity and acceleration vectors which include the effects of the moving axes relative to the fixed axes.)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

The Lagrange Solutions of the Three-Body Problem #8.1

Circular, coplanar orbits with constant angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{i}_\zeta \equiv \omega\,\mathbf{i}_z$

1. **Two bodies of equal mass $m_1 = m_2 = m$ separated by the distance r**

$$m\omega^2\frac{r}{2} = \frac{Gmm}{r^2} \qquad \Longrightarrow \qquad \omega^2 = \frac{2Gm}{r^3}$$

2. **Two bodies m_1 at a distance ρ and m_2 at a distance $r-\rho$ from the center of rotation**

$$\begin{aligned} m_1\omega^2\rho &= \frac{Gm_1m_2}{r^2} \\ m_2\omega^2(r-\rho) &= \frac{Gm_1m_2}{r^2} \end{aligned} \qquad \Longrightarrow \qquad \omega^2 = \frac{G(m_1+m_2)}{r^3}$$

3. **Three bodies of equal mass $m_1 = m_2 = m_3 = m$ at the corners of an equilateral triangle with sides r**

$$m\omega^2\,\frac{r}{2}\sec 30^\circ = \left(\frac{Gmm}{r^2} + \frac{Gmm}{r^2}\right)\cos 30^\circ \qquad \Longrightarrow \qquad \omega^2 = \frac{3Gm}{r^3}$$

4. **Three bodies m_1, m_2, m_3 at the corners of an equilateral triangle with sides r**

$$m_1\omega^2(\mathbf{r}_1 - \mathbf{r}_{cm}) + G\frac{m_1m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1m_3}{r^3}(\mathbf{r}_3 - \mathbf{r}_1) = \mathbf{0}$$

$$m_2\omega^2(\mathbf{r}_2 - \mathbf{r}_{cm}) + G\frac{m_2m_1}{r^3}(\mathbf{r}_1 - \mathbf{r}_2) + G\frac{m_2m_3}{r^3}(\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{0}$$

$$m_3\omega^2(\mathbf{r}_3 - \mathbf{r}_{cm}) + G\frac{m_3m_1}{r^3}(\mathbf{r}_1 - \mathbf{r}_3) + G\frac{m_3m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_3) = \mathbf{0}$$

Add the three equations to obtain

$$\boxed{\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}}$$

Let the origin of coordinates be at the center of mass. Then $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = \mathbf{0}$.

$$\begin{aligned}
[\omega^2 r^3 - G(m_2 + m_3)]\mathbf{r}_1 + Gm_2\mathbf{r}_2 + Gm_3\mathbf{r}_3 &= \mathbf{0} \\
[\omega^2 r^3 - G(m_1 + m_3)]\mathbf{r}_2 + Gm_1\mathbf{r}_1 + Gm_3\mathbf{r}_3 &= \mathbf{0} \\
[\omega^2 r^3 - G(m_1 + m_2)]\mathbf{r}_3 + Gm_1\mathbf{r}_1 + Gm_2\mathbf{r}_2 &= \mathbf{0}
\end{aligned}$$

or

$$\begin{aligned}
[\omega^2 r^3 - G(m_1 + m_2 + m_3)]\mathbf{r}_1 &= \mathbf{0} \\
[\omega^2 r^3 - G(m_1 + m_2 + m_3)]\mathbf{r}_2 &= \mathbf{0} \\
[\omega^2 r^3 - G(m_1 + m_2 + m_3)]\mathbf{r}_3 &= \mathbf{0}
\end{aligned}$$

Hence

$$\omega^2 = \frac{G(m_1 + m_2 + m_3)}{r^3}$$

5. Three collinear masses m_1, m_2, m_3 on ξ axis with

$$\mathbf{r}_1 = r\,\mathbf{i}_\xi \qquad \mathbf{r}_2 = (r + \rho)\,\mathbf{i}_\xi \qquad \mathbf{r}_3 = (r + \rho + \rho\chi)\,\mathbf{i}_\xi$$

The force balance equations are

$$\begin{aligned}
m_1\omega^2 r + \frac{Gm_1m_2}{\rho^2} + \frac{Gm_1m_3}{\rho^2(1+\chi)^2} &= 0 \\
m_2\omega^2(r+\rho) - \frac{Gm_2m_1}{\rho^2} + \frac{Gm_2m_3}{\rho^2\chi^2} &= 0 \\
m_3\omega^2(r+\rho+\rho\chi) - \frac{Gm_3m_1}{\rho^2(1+\chi)^2} - \frac{Gm_3m_2}{\rho^2\chi^2} &= 0
\end{aligned}$$

Replace last equation by sum of three equations $m_1 r + m_2(r+\rho) + m_3(r+\rho+\rho\chi) = 0$. Then

$$\begin{aligned}
m_1\omega^2 r + \frac{Gm_1m_2}{\rho^2} + \frac{Gm_1m_3}{\rho^2(1+\chi)^2} &= 0 \\
m_2\omega^2(r+\rho) - \frac{Gm_2m_1}{\rho^2} + \frac{Gm_2m_3}{\rho^2\chi^2} &= 0 \\
m_1 r + m_2(r+\rho) + m_3[r + \rho(1+\chi)] &= 0
\end{aligned}$$

Third equation $\Longrightarrow$ $r = -\rho\dfrac{m_2 + (1+\chi)m_3}{m_1 + m_2 + m_3}$

First equation $\Longrightarrow$ $\omega^2 = \dfrac{G(m_1 + m_2 + m_3)}{\rho^3(1+\chi)^2}\,\dfrac{m_2(1+\chi)^2 + m_3}{m_2 + (1+\chi)m_3}$

Second equation $\Longrightarrow$

$$\begin{aligned}
&(m_1 + m_2)\chi^5 + (3m_1 + 2m_2)\chi^4 + (3m_1 + m_2)\chi^3 \\
&\quad - (m_2 + 3m_3)\chi^2 - (2m_2 + 3m_3)\chi - (m_2 + m_3) = 0
\end{aligned}$$