16.346 Astrodynamics
Fall 2008

Lagrange's equations for the boundary-value problem

$$\sqrt{\mu}\,(t_2 - t_1) = 2a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi) \tag{1}$$

$$r_1 + r_2 = 2a(1 - \cos\psi\cos\phi) \tag{2}$$

$$\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta = a(\cos\psi - \cos\phi) \tag{3}$$

Gauss's Equation for the Semimajor Axis

Eliminate $\cos\phi$ between (2) and (3):

$$\frac{1}{a} = \frac{2\sin^2\psi}{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta\cos\psi}$$

When ψ and θ are very small (of the order of 2 or 3 degrees), then r_1 and r_2 will have almost the same value. The denominator will be determined as the difference between two almost equal terms resulting in a severe loss of accuracy. To prevent this, Gauss wrote

$$\frac{1}{a} = \frac{\sin^2\psi}{2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta\,(\ell + \sin^2\frac{1}{2}\psi)} \tag{4}$$

where ℓ is defined as

$$\boxed{\ell = \frac{\sqrt{\dfrac{r_2}{r_1}} + \sqrt{\dfrac{r_1}{r_2}}}{4\cos\frac{1}{2}\theta} - \frac{1}{2}}$$

The problem of subtracting two almost equal quantities still exists but Gauss had a different method for calculating ℓ which avoided any subtraction:

$$\ell = \frac{\sin^2\frac{1}{4}\theta + \tan^2 2\omega}{\cos\frac{1}{2}\theta} \quad \text{where} \quad \tan(\tfrac{1}{4}\pi + \omega) = \left(\frac{r_2}{r_1}\right)^{\frac{1}{4}}$$

An alternate method, which does not require any inverse trigonometric function is to express

$$r_2 = r_1(1 + \epsilon)$$

The quantity ϵ is simply the fractional part resulting when r_2 is divided by r_1 (assuming, of course, that r_2 exceeds r_1). The result is

$$\tan^2 2\omega = \frac{\frac{1}{4}\epsilon^2}{\sqrt{\dfrac{r_2}{r_1}} + \dfrac{r_2}{r_1}\left(2 + \sqrt{\dfrac{r_2}{r_1}}\right)}$$

Gauss's Time Equation

Eliminate $\cos\phi$ between (1) and (3):

$$\sqrt{\frac{\mu}{a^3}}\,(t_2 - t_1) = 2\psi - \sin 2\psi + \frac{2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{a}\sin\psi \tag{5}$$

Next using Eq. (4) for $1/a$ he obtained

$$\frac{\sqrt{\mu}(t_2 - t_1)\sin^3\psi}{(2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta)^{\frac{3}{2}}(\ell + \sin^2\frac{1}{2}\psi)^{\frac{3}{2}}} \equiv \frac{m\sin^3\psi}{(\ell + \sin^2\frac{1}{2}\psi)^{\frac{3}{2}}} = 2\psi - \sin 2\psi + \frac{\sin^3\psi}{\ell + \sin^2\frac{1}{2}\psi}$$

where he defined

$$\boxed{m = \frac{\sqrt{\mu}(t_2 - t_1)}{(2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta)^{\frac{3}{2}}}}$$

which requires that $0 < \theta < 180°$.

Finally, Gauss defined

$$\boxed{y^2 = \frac{m^2}{\ell + \sin^2\frac{1}{2}\psi}}$$

so that the time equation (5) can be written as

$$m \times \frac{y^3}{m^3} = \frac{2\psi - \sin 2\psi}{\sin^3\psi} + \frac{y^2}{m^2} \qquad \text{or} \qquad \boxed{y^3 - y^2 = m^2\,\frac{2\psi - \sin 2\psi}{\sin^3\psi}}$$

which are Gauss' equations, to be solved for y and ψ.

The Orbital Parameter and the Significance of y

From Lecture 9 on Page 3

$$p = \frac{\sin\phi}{\sin\psi}\,p_m = \frac{\sin\phi}{\sin\psi} \times \frac{2r_1 r_2 \sin^2\frac{1}{2}\theta}{c} = \frac{\sin\phi}{\sin\psi} \times \frac{2r_1 r_2 \sin^2\frac{1}{2}\theta}{2a\sin\psi\sin\phi} = \frac{r_1 r_2 \sin^2\frac{1}{2}\theta}{a\sin^2\psi}$$

Then from

$$a = \frac{2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta\,(\ell + \sin^2\frac{1}{2}\psi)}{\sin^2\psi} \qquad \text{and} \qquad y^2 = \frac{m^2}{\ell + \sin^2\frac{1}{2}\psi}$$

we have

$$a\sin^2\psi = \frac{2m^2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{y^2} \qquad \text{where} \qquad m^2 = \frac{\mu(t_2 - t_1)^2}{(2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta)^3}$$

so that

$$\boxed{p = \frac{r_1^2 r_2^2 y^2 \sin^2\theta}{\mu(t_2 - t_1)^2}} = \frac{h^2}{\mu}$$

from which

$$\frac{\frac{1}{2}h(t_2 - t_1)}{\frac{1}{2}r_1 r_2 \sin\theta} = \boxed{y = \frac{\text{Area of sector}}{\text{Area of triangle}}}$$

Changing the independent variable from ψ to $x = \sin^2 \frac{1}{2}\psi$

Define

$$\boxed{Q = \frac{2\psi - \sin 2\psi}{\sin^3 \psi}}$$

then

$$3Q \sin^2 \psi \cos \psi + \sin^3 \psi \frac{dQ}{d\psi} = 2 - 2\cos 2\psi = 4 \sin^2 \psi$$

Now

$$\frac{dx}{d\psi} = \sin \tfrac{1}{2}\psi \cos \tfrac{1}{2}\psi = \frac{1}{2}\sin \psi$$

so that

$$3Q \cos \psi + \frac{1}{2}\sin^2 \psi \frac{dQ}{dx} = 4$$

Since

$$\cos \psi = 1 - 2\sin^2 \tfrac{1}{2}\psi = 1 - 2x$$

$$\sin^2 \psi = 4\sin^2 \tfrac{1}{2}\psi(1 - \sin^2 \tfrac{1}{2}\psi) = 4x(1 - x)$$

then

$$2x(1-x)\frac{dQ}{dx} = 4 - (3 - 6x)Q$$

Write

$$Q = \frac{4}{3}(1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + \cdots)$$

Substitute and equate like powers of x to obtain:

$$q_1 = \frac{6}{5} \qquad q_2 = \frac{8}{7}q_1 \qquad q_3 = \frac{10}{9}q_2 \qquad q_4 = \frac{12}{11}q_3 \qquad \text{etc.}$$

resulting in

$$Q(x) = \frac{4}{3}F(3,1;\tfrac{5}{2};x) = \frac{4}{3}\left(1 + \frac{6}{5}x + \frac{6\cdot 8}{5\cdot 7}x^2 + \frac{6\cdot 8\cdot 10}{5\cdot 7\cdot 9}x^3 + \frac{6\cdot 8\cdot 10\cdot 12}{5\cdot 7\cdot 9\cdot 11}x^4 + \cdots\right)$$

$$F(3,1;\tfrac{5}{2};x) = \cfrac{1}{1 - \cfrac{\gamma_1 x}{1 - \cfrac{\gamma_2 x}{1 - \cfrac{\gamma_3 x}{1 - \ddots}}}} \qquad \gamma_n = \begin{cases} \dfrac{(n+2)(n+5)}{(2n+1)(2n+3)} & n \text{ odd} \\[2ex] \dfrac{n(n-3)}{(2n+1)(2n+3)} & n \text{ even} \end{cases}$$

a. The series converges for $-1 < x < 1$.

b. The continued fraction converges for $-\infty < x < 1$.

Note: The function $F(\alpha, \beta; \gamma; x)$ is **Gauss' Hypergeometric Function** which we will exam in some detail in the next Lecture.

The Universal Form of Gauss' Method

We can extend the definition of x so that

$$x = \begin{cases} \sin^2 \frac{1}{4}(E_2 - E_1) & \text{ellipse} \\ 0 & \text{parabola} \\ -\sinh^2 \frac{1}{4}(H_2 - H_1) & \text{hyperbola} \end{cases}$$

The range of x is $-\infty < x < 1$. The series representation of $Q(x)$ will not converge when $x < -1$. However the continued fraction does converge over the full range.

Possible Algorithm

Gauss' equations are:

$$y^2 = \frac{m^2}{\ell + x} \quad \text{and} \quad y^3 - y^2 = m^2 Q(x)$$

in terms of x. The following is a recursive algorithm for the solution:

1. Set $x = 0$

2. Solve of cubic $y^3 - y^2 = m^2 Q(x)$

Note: Solution of the cubic: $y = 1 + \frac{4}{3}\sinh^2 \frac{1}{3} z$ where $\sinh z = \frac{3}{2}\sqrt{3m^2 Q}$

3. Obtain new x from

$$x = \frac{m^2}{y^2} - \ell$$

and repeat until the process converges.

Gauss' Successive Substitution Algorithm $0 < \theta < \pi$

1. Given $r_1,\ r_2,\ \theta,\ \sqrt{\mu}\,(t_2 - t_1)$

2. Compute $\ell = \dfrac{r_1 + r_2}{4\sqrt{r_1 r_2}\cos\frac{1}{2}\theta} - \dfrac{1}{2}$ and $m^2 = \dfrac{\mu(t_2 - t_1)^2}{(2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta)^3}$

3. Initialize $x = 0$

4. Calculate

$$\xi(x) = \cfrac{\frac{2}{35}x^2}{1 + \frac{2}{35}x - \cfrac{\frac{40}{63}x}{1 - \cfrac{\frac{4}{99}x}{1 - \cfrac{\frac{70}{143}x}{1 - \cfrac{\frac{18}{195}x}{1 - \cfrac{\frac{108}{255}x}{1 - \ddots}}}}}}$$

and $h = \dfrac{m^2}{\frac{5}{6} + \ell + \xi(x)}$

5. Solve the cubic $y^3 - y^2 - hy - \dfrac{h}{9} = 0$

6. Determine new $x = \dfrac{m^2}{y^2} - \ell$ and repeat until x no longer changes.

7. Calculate the orbital elements:

$$\frac{1}{a} = \frac{8 r_1 r_2 y^2 x(1 - x)(1 + \cos\theta)}{\mu(t_2 - t_1)^2} \qquad p = \frac{r_1^2 r_2^2 y^2 \sin^2\theta}{\mu(t_2 - t_1)^2}$$

Avoiding the Continued Fraction When ψ is Not Small

Instead of the continued fraction (which we shall learn more about in the next lecture), we can use the closed form expression

$$\xi(\psi) = \frac{\sin^3\psi - \frac{3}{4}(2\psi - \sin 2\psi)(1 - \frac{6}{5}\sin^2\frac{1}{2}\psi)}{\frac{9}{10}(2\psi - \sin 2\psi)}$$

Since $x = \sin^2\frac{1}{2}\psi$, then $\psi = 2\arcsin(\sqrt{x}\,)$.

"The numerator of this expression is a quantity of the seventh order, the denominator of the third order, and ξ, therefore, of the fourth order, if ψ is regarded as a quantity of the first order. Hence it is inferred that this formula is not suited to the exact numerical computation of ξ when ψ does not denote a very considerable angle."

Karl Friedrich Gauss

Solving the Cubic Equation Pages 321 & 54

The solution of the cubic equation

$$y^3 - y^2 - hy - \frac{1}{9}h = 0$$

using the method developed on **Page 321** of your textbook, is

$$y = \frac{1}{3}(1 + w\sqrt{1+3h}\,)$$

where w is the solution of

$$w^3 - 3w = 2\frac{1+6h}{(1+3h)^{\frac{3}{2}}} = 2b$$

Note: Barker's Equation is $w^3 + 3w = 2b$

We must address the cases $b < 1$ and $b \geq 1$ separately:

$b < 1$ Write $w = 2\cos\frac{2}{3}x = 2(1 - 2\sin^2\frac{1}{3}x)$ and $b = \cos 2x = 1 - 2\sin^2 x$

Then the cubic equation becomes $4\cos^2\frac{2}{3}x = 3\cos\frac{2}{3}x = \cos 2x$ which is an identity for cosine functions. Hence:

$$w = 2\cos(\frac{1}{3}\arccos b)$$

$b \geq 1$ Define $w = 2\cosh\frac{2}{3}x = 2(1 + 2\sinh^2 x)$ and $b = \cosh 2x = 1 + 2\sinh^2 x$

Then the cubic equation becomes $4\cosh^2\frac{2}{3}x - 3\cosh\frac{2}{3}x = \cosh 2x$ which is an identity for hyperbolic cosines. Hence:

$$w = 2\cosh(\frac{1}{3}\operatorname{arccosh} b)$$