MIT OpenCourseWare
http://ocw.mit.edu

16.323 Principles of Optimal Control
Spring 2008

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.

16.323 Homework Assignment #2

**Any code for this homework should be submitted online using the class web
page - do not forget, as it will be graded as part of your solution.**

1. Using dynamic programming, find the minimum-time path and time from point A to
   point B, moving only to the right. The time to travel between points is shown on each
   segment. Compare the DP result (path and cost) with the "forward greedy" path (i.e.
   the one where you just decide the best path of the options available as you move from
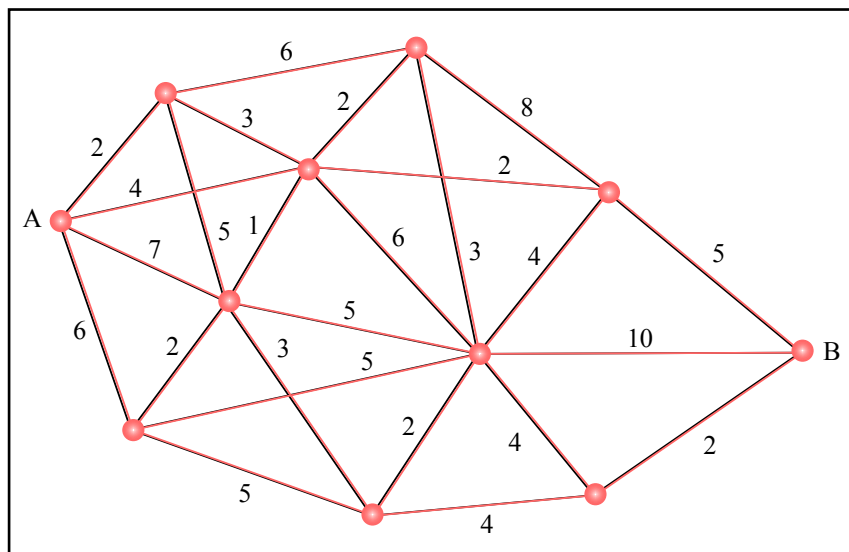   A to B assuming that you also only move to the right to avoid repeated nodes).

Figure by MIT OpenCourseWare.

2. The discrete approximation to a nonlinear continuously operating system is given by

$$x_{k+1} = x_k - 0.4x_k^2 + u_k$$

   The state and control values are constrained by

$$0.0 \leq \ x_k \ \leq 1.0$$
$$-0.4 \leq \ u_k \ \leq 0.4$$

   Quantize the state into levels 0.0, 0.5, 1.0, and the control into levels -0.4, -0.2, 0.0,
   0.2, 0.4. The performance measure to be minimized is

$$J = 4|x_2| + \sum_{k=0}^{1} |u_k|$$

(a) Use dynamic programming with linear interpolation to find the optimal control law and the associated cost for all possible values of x_0 and x_1.

(b) From the above results find the optimal control sequence $\{u_0^*, u_1^*\}$ and the minimum cost if the initial state is 1.0.

(c) Implement the dynamic programming algorithm in MATLAB to confirm your answers. Use your MATLAB code to explore how the quantization levels of x_k and u_k affect the answer.

3. For the first-order discrete system $x_{k+1} = 1.05x_k + u_k$. The performance measure to be minimized is
$$J = x(N)^2 + \frac{1}{2}\sum_{k=0}^{N-1} qx^2(k) + ru^2(k)$$

(a) For the case $N \to \infty$, solve the discrete equation for the steady state P and gain F as a function of q and r.

(b) For $q = r = 1$, confirm that the backwards recursion agrees with this steady state prediction. Based on these results, how large do you think N needs to be before we can ignore the transient?

(c) Compare the performance of the optimal controller ($q = r = 1$) with the ad-hoc design $u_k = -0.4x_k$.

4. Given the optimal control problem for a scalar nonlinear system:
$$\dot{x} = xu \qquad x(0) = 1$$
$$J(u) = x(1)^2 + \int_0^1 (x(t)u(t))^2 dt$$

find the optimal feedback strategy by solving the associated HJB equation. Hint: show that the HJB differential equation admits a solution of the form $J^\star = p(t)x(t)^2$.

5. Use the matrix inversion lemma to show that the discrete algebraic Riccati equation of the form:
$$P_{ss} = Q + A^T \left\{ P_{ss} - P_{ss}B \left[R + B^T P_{ss}B \right]^{-1} B^T P_{ss} \right\} A \qquad (1)$$

can be rewritten in the slightly cleaner form:
$$P_{ss} = Q + A^T P_{ss} \left[I + BR^{-1}B^T P_{ss} \right]^{-1} A \qquad (2)$$

Use this form to analytically solve for the steady state $P_{ss} = [P_{11} \ P_{12}; \ P_{21} \ P_{22}]$ if
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 1$$

Note that A is invertible in this case, which enables further simplification in solving for P_{ss}. Confirm your result using Matlab.

6. So who was Riccati and what was his contribution to control theory?